Number theoretic properties of generating functions related to Dyson's rank for partitions into distinct parts.

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- $p(n)$ is the number of partitions of n.
- $Q(n)$ is the number of partitions of n into distinct parts.

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- Are there combinatorial explanations for these elegant identities?

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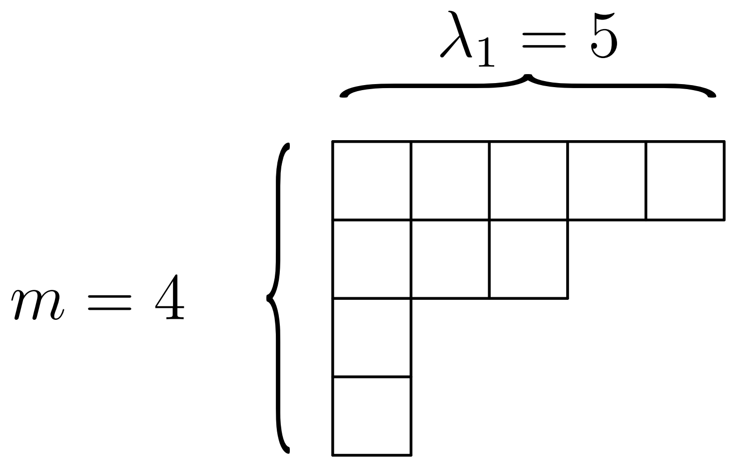

Combinatorial intepretations

- Atkin and Swinnerton-Dyer: When the partitions of $5n + 4$ are sorted by their rank modulo 5, the resulting 5 sets all have the same number of elements!

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- Taken modulo 7, the rank also sorts the partitions of $7n+5$ into 7 equal-sized groups.
- Failed to explain $p(11n+6) \equiv 0 \pmod{11}$. Garvan discovered the *crank*, which explained this identity along with many other congruences.

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- Can a rank or similar combinatorial invariant be used to explain congruences for $Q(n)$?

- The rank provides a combinatorial interpretation for $j = 1$ and $j = 2$!

Theorem (M.). Define $T(m, k; n)$ to be the number of partitions of n into distinct parts having rank congruent to $m \pmod{k}$. Then

$$T(0,4;n) = T(1,4;n) = T(2,4;n) = T(3,4;n)$$

if and only if $24n + 1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$ such that the largest power of p dividing $24n + 1$ is p^e where e is odd.

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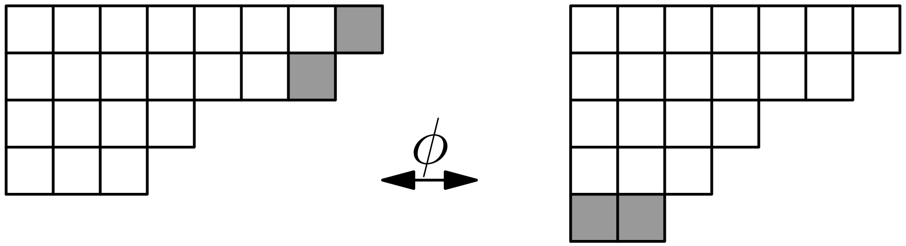

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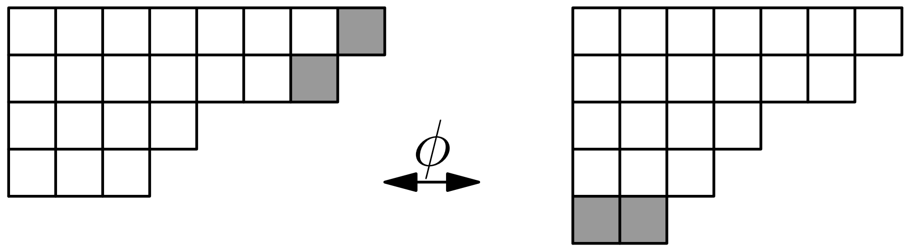

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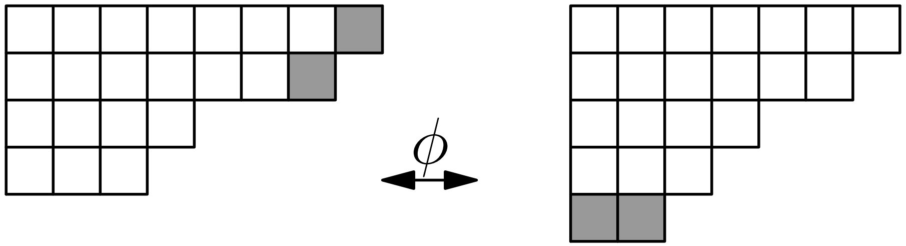

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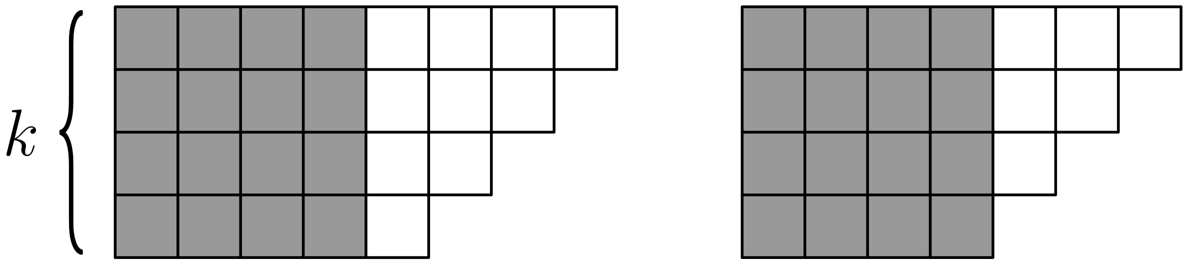

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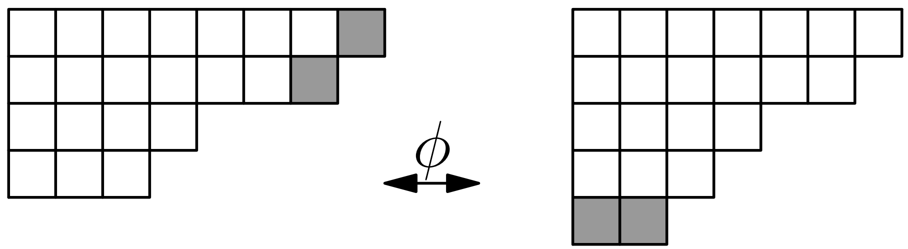

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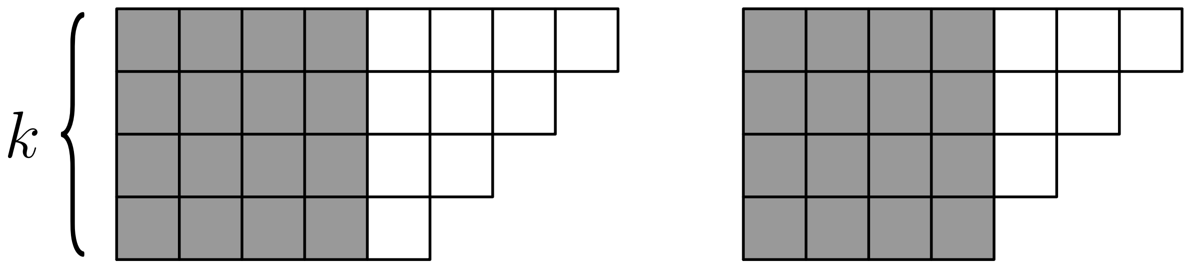

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- Andrews, Dyson, Hickerson: $T(0,2;n) = T(1,2;n)$ if and only if $24n+1$ has a prime divisor $p \not\equiv \pm 1 \pmod{24}$ such that the largest power of p dividing $24n+1$ is p^e for some odd positive integer e.

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- Thus $T(0, 4; n) = T(1, 4; n) = T(2, 4; n) = T(3, 4; n)$ for such n, and the set of such n is dense in the integers. Thus $Q(n)$ is nearly always divisible by 4.

Generating functions

- Let $Q(n,r)$ denote the number of partitions of n into distinct parts having rank r, and define

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$$G(z, q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{(1 - zq)(1 - zq^2) \cdots (1 - zq^s)}$$

for $z, q \in \mathbb{C}$ with $|z| \leq 1$, $|q| < 1$.

$G(z,q)$ at fourth roots of unity z

Theorem (M.). Let $q \in \mathbb{C}$ with $|q| < 1$. Then

$$G(i,q) = \sum_{k=0}^{\infty} i^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} i^{k-1} q^{k(3k-1)/2}$$

$$G(-i,q) = \sum_{k=0}^{\infty} (-i)^k q^{k(3k+1)/2} + \sum_{k=1}^{\infty} (-i)^{k-1} q^{k(3k-1)/2}$$

- $G(1,q) = \sum_{n=0}^{\infty} Q(n) q^n = (1+q)(1+q^2)(1+q^3)\cdots$ is a weight 0 modular form, in the variable τ where $q = e^{2\pi i \tau}$.

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A new false theta function (or two)

- It follows that

$$qG(i, q^{24}) = \sum_{k=0}^{\infty} i^k q^{(6k+1)^2} + \sum_{k=1}^{\infty} i^{k-1} q^{(6k-1)^2}$$

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- Not true theta functions, but they resemble theta functions in the sense that their coefficients are roots of unity and are 0 whenever the exponent of q is not a perfect square. Such functions are known as *false theta functions*.

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- $R(-1, q)$ is one of Ramanujan's famous "mock theta functions".

The relation between G and R

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or alternatively,

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- The analytic behavior of the false theta functions $G(\pm i,q)$ gives the behavior of $R(\pm i,q)$ for q outside the unit disk!

Relation to modular forms

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- Therefore, the functions $G(\pm i, q^{-1})$ appear naturally within the theory of automorphic forms.
- What about $G(w, q)$, and $G(w, q^{-1})$, for other roots of unity w?

Relating $G(w,q)$ to modular forms

- Define the series

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Theorem (M., Ono)**.** We have

$$q^{\frac{1}{12}} \cdot D(\zeta;q)D(\zeta^{-1};q) = 4 \cdot \frac{\eta(2\tau)^4}{\eta(\tau)^2\eta(\zeta^2;2\tau)}$$

is a weight 1 modular form for roots of unity $\zeta \neq \pm 1$.

The function $\widehat{G}(w, q)$

- Define $\widehat{G}(w,q) = G(w, q^{-1})$. Formal manipulation yields

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Theorem (M., Ono)**.** Suppose that $-w^{-1} \neq 1$ is an mth primitive root of unity. If $0 \leq r < m$, then $\lim_{n\to\infty} \widehat{G}_{mn+r}(w;q)$ is a well defined q-series and satisfies

$$\lim_{n\to\infty} \widehat{G}_{mn+r}(w;q) = \lim_{n\to\infty} \widehat{G}_{mn}(w;q) + \frac{(-w^{-1})^r - 1}{w+1} \frac{1}{(w^{-1}q;q)_\infty}.$$

Example: The case $-w^{-1} = -1$

$$
\begin{aligned}
\widehat{G}_1(1;q) &= -q - q^2 - q^3 - q^4 - q^5 - q^6 - q^7 - q^8 - q^9 - \cdots \\
\widehat{G}_3(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 3q^5 - 4q^6 - 5q^7 - 6q^8 - 8q^9 - \cdots \\
\widehat{G}_5(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 7q^7 - 9q^8 - 13q^9 - \cdots \\
\widehat{G}_7(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - 15q^9 - \cdots \\
\widehat{G}_9(1;q) &= -q - q^2 - 2q^3 - 2q^4 - 4q^5 - 5q^6 - 8q^7 - 10q^8 - 16q^9 - \cdots
\end{aligned}
$$

and

$$
\begin{aligned}
\widehat{G}_2(1;q) &= 1 + q^2 + q^3 + 2q^4 + 2q^5 + 3q^6 + 3q^7 + 4q^8 + 4q^9 + \cdots \\
\widehat{G}_4(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 5q^6 + 6q^7 + 9q^8 + 10q^9 + \cdots \\
\widehat{G}_6(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 11q^8 + 13q^9 + \cdots \\
\widehat{G}_8(1;q) &= 1 + q^2 + q^3 + 3q^4 + 3q^5 + 6q^6 + 7q^7 + 12q^8 + 14q^9 + \cdots
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- For $-w^{-1}$ a primitive mth root of unity, it now makes sense to define the q-series

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- Also define $\widehat{D}(w, q) = (1 + w^{-1})\widehat{G}(w, q) + (1 - w^{-2})(\psi(-w^2, q) - 1)$.

Relating $\widehat{G}(w,q)$ to modular forms

Theorem (Folsom)**.** Let $-\omega^{-1} \neq 1$ be a primitive mth root of unity. Then $q^{-1/12}\widehat{D}(\omega,q)\widehat{D}(\omega^{-1},q)$ is the weight 1 modular form

$$q^{-1/12}\widehat{D}(\omega,q)\widehat{D}(\omega^{-1}q) = \frac{\eta^4(q^2)\eta^2(\omega^2,q)}{\eta^2(q)\eta^3(\omega^2,q^2)}$$

where $\eta(\omega,q) = q^{1/12}(\omega q;q)_\infty(\omega^{-1}q;q)_\infty$.

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where $\eta(\omega,q) = q^{1/12}(\omega q;q)_\infty(\omega^{-1}q;q)_\infty$.

- Thus G and $\widehat{G}$ appear naturally within the theory of automorphic forms!

Observations and Future Work

- The rank fails to explain the divisibility of $Q(n)$ by higher powers of 2. Is there a generalization of the rank that can be used to divide the partitions of $Q(n)$ into m equal-sized groups whenever $Q(n)$ is divisible by m for any positive integer m?

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- Are there other partition functions for which we can obtain congruences via the rank or related combinatorial invariants?

Observations and Future Work

- The rank fails to explain the divisibility of $Q(n)$ by higher powers of 2. Is there a generalization of the rank that can be used to divide the partitions of $Q(n)$ into m equal-sized groups whenever $Q(n)$ is divisible by m for any positive integer m?
- Are there other partition functions for which we can obtain congruences via the rank or related combinatorial invariants?
- We have seen that $G(z,q)$ and $R(z,q)$ are related at $z = \pm i$. Are these the only values of z for which they are related in some way?

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