On the distribution of arithmetic sequences in the Collatz graph

Keenan Monks, Harvard University
Ken G. Monks, University of Scranton
Ken M. Monks, Colorado State University
Maria Monks, UC Berkeley

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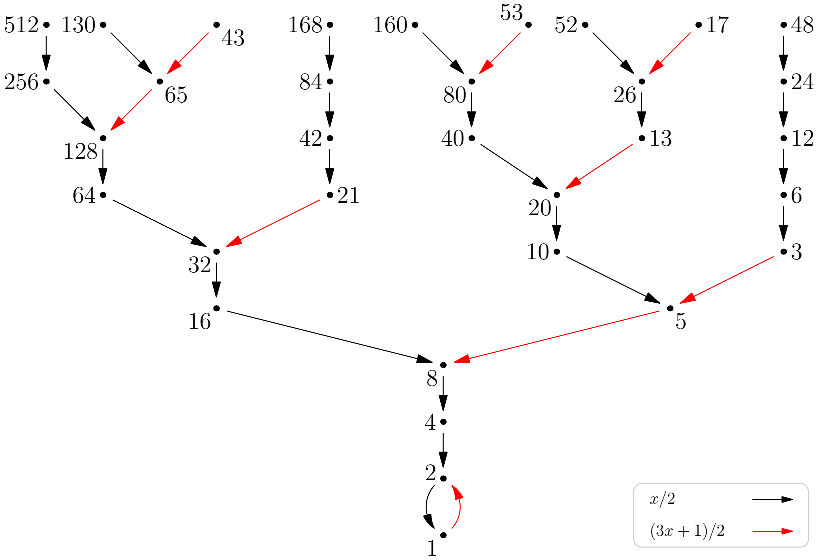
The Collatz graph $\mathcal{G}$



Two smaller conjectures

- **The Nontrivial Cycles conjecture:** There are no T-cycles of positive integers other than the cycle $\overline{1, 2}$.
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- Both still unsolved.

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- **Theorem.** (K. M. Monks, 2006.) Every arithmetic sequence is sufficient.
- In fact, Monks shows that every positive integer relatively prime to 3 can be *back-traced* to an element of a given arithmetic sequence.
- Every integer congruent to 0 mod 3 *forward-traces* to an integer relatively prime to 3, at which point the orbit contains no more multiples of 3.

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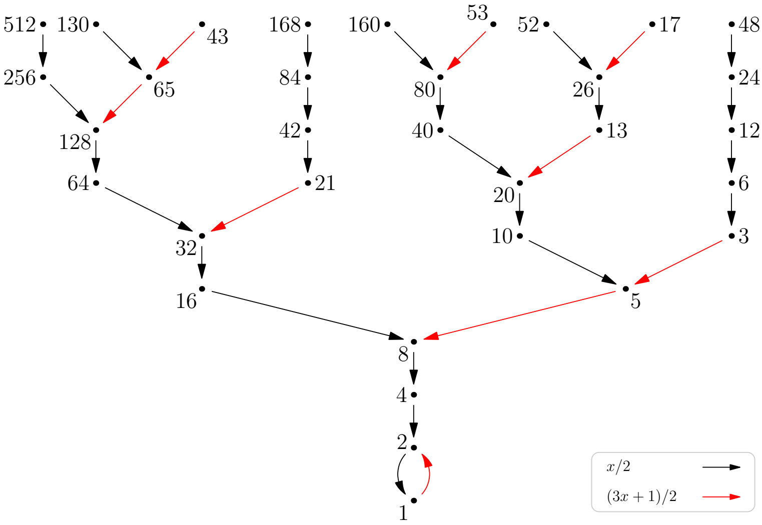
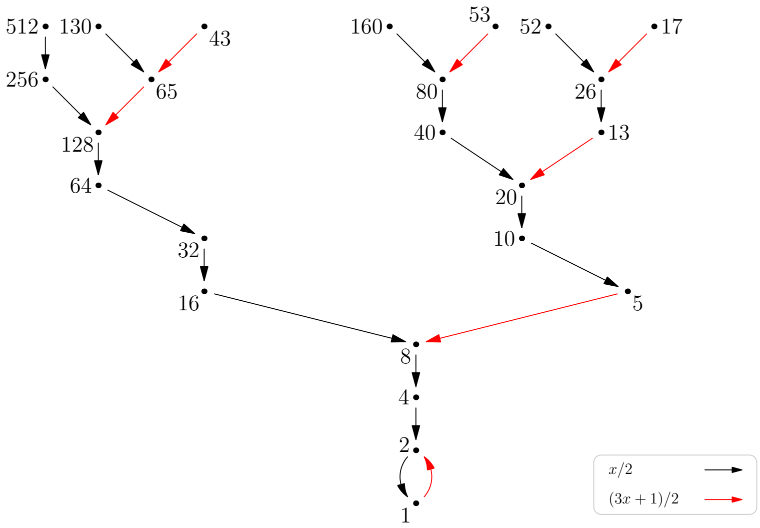


The pruned Collatz graph $\widetilde{\mathcal{G}}$



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4. In which infinite back-tracing paths does a given arithmetic sequence $\{a + bN\}$ occur?

Attempting the first question

A family of sparse sufficient sets

Proposition (Monks, Monks, Monks, M.)

For any function $f : \mathbb{N} \to \mathbb{N}$ *and any positive integers* a *and* b*,*

$$\{2^{f(n)}(a + bn) \mid n \in \mathbb{N}\}$$

is a sufficient set.

Proof.

Any positive integer x merges with some number of the form $a + bN$. Then $2^{f(N)}(a + bN)$, which maps to $a + bN$ after $f(N)$ iterations of T, also merges with x. □

Corollary

For any fixed a *and* b*, the sequence* $(a + bn) \cdot 2^n$ *is a sufficient set with asymptotic density zero in the positive integers.*

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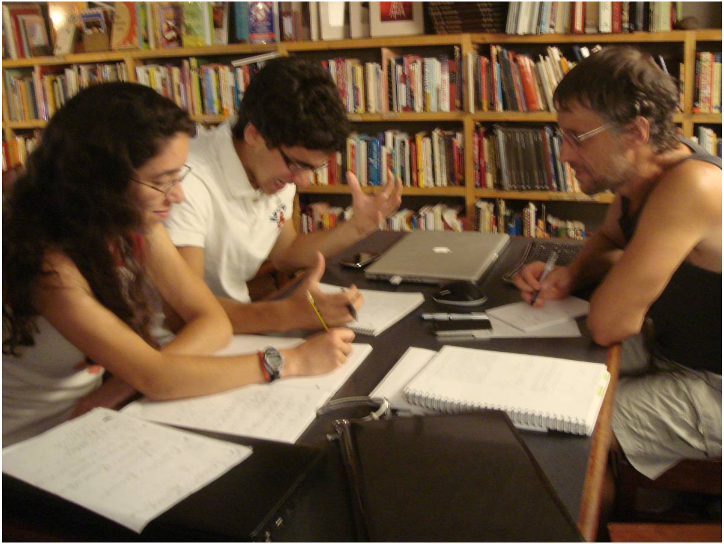
Attempting the second question



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- Define the *length* of a finite back-tracing path to be the number of red arrows in the path.
- Want to find the shortest back-tracing path to an element of the arithmetic sequence $a \bmod b$ for various a and b.
- Consider three cases: when b is a power of 2, a power of 3, or relatively prime to 2 and 3.

Efficient back-tracing

Proposition

Let $b \in \mathbb{N}$ *with* $\gcd(b, 6) = 1$*, and let* $a < b$ *be a nonnegative integer. Let* e *be the order of* $\frac{3}{2}$ *modulo* b*. Then any* $x \in \mathbb{N} \setminus 3\mathbb{N}$ *can be back-traced to an integer congruent to* a *mod* b *via a path of length at most* $(b-1)e$*.*

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Proposition

Let $n \geq 1$ *and* $a < 2^n$ *be nonnegative integers. Then any* $x \in \mathbb{N} \setminus 3\mathbb{N}$ *can be back-traced to an integer congruent to* a *mod* 2^n *using a path of length at most* $\lfloor \log_2 a + 1 \rfloor$*.*

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Let $m \geq 1$ and $a < 3^m$ be nonnegative integers. Then any $x \in \mathbb{N} \setminus 3\mathbb{N}$ can be back-traced to infinitely many odd elements of $a + 3^m\mathbb{N}$ via an admissible sequence of length 1.

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Proposition

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   Pretty close, depending on b.
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Attempting the third question

Infinite back-tracing

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- We think of an infinite back-tracing parity vector as an element of $\mathbb{Z}_2$, the ring of 2-adic integers.
- Some are simple to describe: those that end in $\overline{0}$. These are the positive integers $\mathbb{N} \subset \mathbb{Z}_2$.

Infinite back-tracing

- An *infinite back-tracing sequence* is a sequence of the form

$$x_0, x_1, x_2, \ldots$$

  for which $T(x_i) = x_{i-1}$ for all $i \geq 1$.
- An *infinite back-tracing parity vector* is the binary sequence formed by taking an infinite back-tracing sequence mod 2.
- We think of an infinite back-tracing parity vector as an element of $\mathbb{Z}_2$, the ring of 2-adic integers.
- Some are simple to describe: those that end in $\overline{0}$. These are the positive integers $\mathbb{N} \subset \mathbb{Z}_2$.
- When there are infinitely many 1's, they are much harder to describe.

Uniqueness of infinite back-tracing vectors

Proposition

Let $x \in \mathbb{N} \setminus 3\mathbb{N}$*, and suppose* v *is a back-tracing parity vector for* x *containing infinitely many* 1*'s. If* v *is also a back-tracing parity vector for* y*, then* $x = y$*.*

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- (Bernstein, 1994.) This gives a map $\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$ that sends v to the unique 2-adic whose T-orbit, taken mod 2, is v.
- Similarly, we can define a map $\Psi : \mathbb{Z}_2 \setminus N \to \mathbb{Z}_3$ that sends v to the unique 3-adic having v as an infinite back-tracing parity vector.

What are the back-tracing parity vectors starting from positive integers?

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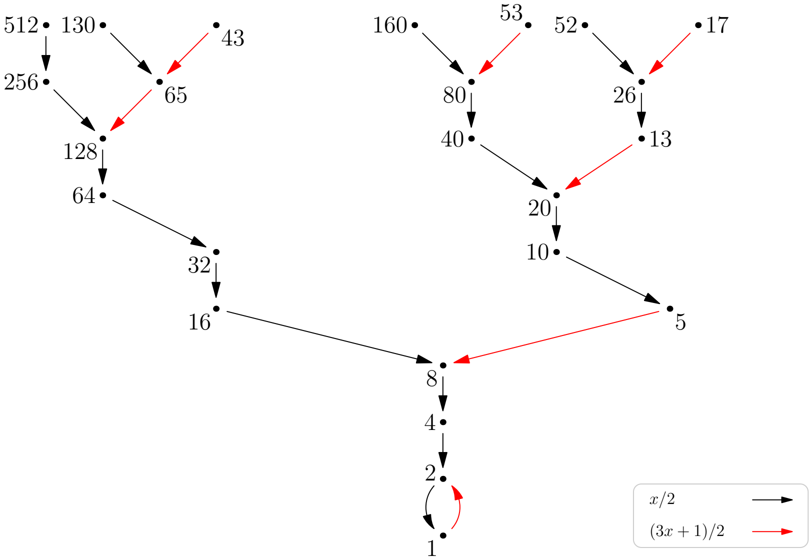
Can we write down an irrational one?
The best we can do is a recursive construction, such as the greedy back-tracing vector that follows red whenever possible. Even this is hard to describe explicitly.

Another look at $\widetilde{\mathcal{G}}$



Natural questions arising from the sufficiency of arithmetic progressions

1. Can we find a sufficient set with asymptotic density 0 in $\mathbb{N}$?
   Yes!
2. For a given $x \in \mathbb{N} \setminus 3\mathbb{N}$, how "close" is the nearest element of $\{a + bN\}_{N \geq 0}$ that we can back-trace to?
   Pretty close, depending on b.
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Attempting the fourth question

Strong sufficiency in the reverse direction

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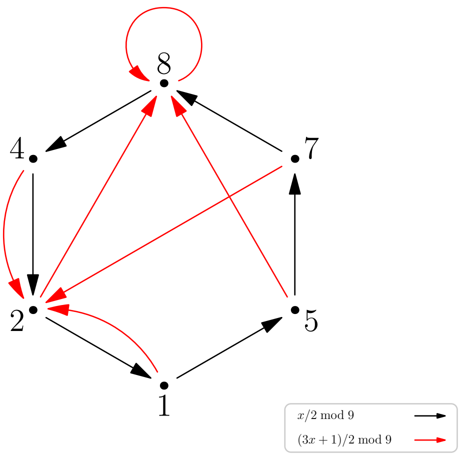
We say that the set of positive integers congruent to 2 mod 9 is *strongly sufficient in the reverse direction*.

Proof by picture: the pruned Collatz graph mod 9.



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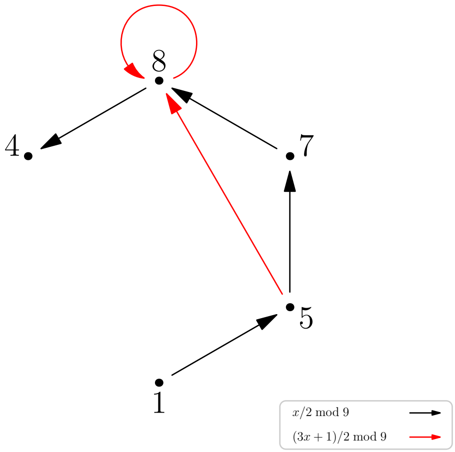

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- S is *strongly sufficient* if it is strongly sufficient in both directions.
- **How this helps:** Suppose we can show that, for instance, the set of integers congruent to 1 mod 2^n is strongly sufficient for every n. Then the nontrivial cycles conjecture is true!

The graphs Γ_k

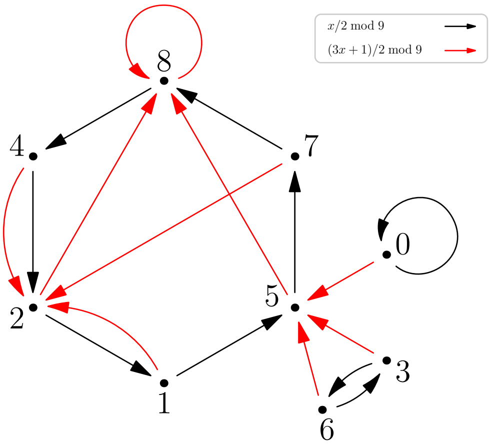
Definition

For $k \in \mathbb{N}$, define Γ_k to be the two-colored directed graph on $\mathbb{Z}/k\mathbb{Z}$ having a **black** arrow from r to s if and only if $\exists x, y \in \mathbb{N}$ with

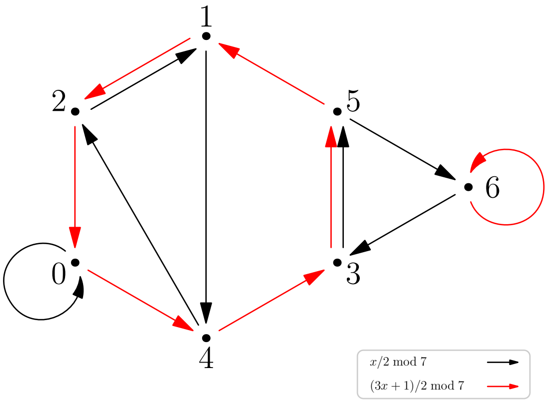
$$x \equiv r \text{ and } y \equiv s \pmod{k}$$

with $x/2 = y$, and a **red** arrow from r to s if there are such an x and y with $(3x+1)/2 = y$.

Example: Γ_9



Example: Γ_7



A criterion for strong sufficiency

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Let $n \in \mathbb{N}$*, and let* $a_1, \ldots, a_k$ *be* k *distinct residues mod* n*.*

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- *Let Γ''_n be the graph formed from Γ'_n by deleting any edge which is not contained in any cycle of Γ'_n.*

If Γ''_n is a disjoint union of cycles and isolated vertices, and each of the cycles have length less than $630,138,897$, then the set

$$a_1, \ldots, a_k \bmod n$$

is strongly sufficient.

A list of strongly sufficient sets					
0 mod 2	1, 4 mod 9	1, 2, 6 mod 7	3, 4, 7 mod 10	2, 7, 8 mod 11	4, 5, 12 mod 14
1 mod 2	1, 8 mod 9	0, 1, 3 mod 8	3, 6, 7 mod 10	3, 4, 5 mod 11	4, 6, 11 mod 14
1 mod 3	4, 5 mod 9	0, 1, 6 mod 8	3, 7, 8 mod 10	3, 4, 8 mod 11	4, 11, 12 mod 14
2 mod 3	4, 7 mod 9	2, 4, 7 mod 8	4, 5, 7 mod 10	3, 4, 9 mod 11	6, 7, 8 mod 14
1 mod 4	5, 8 mod 9	2, 5, 7 mod 8	5, 6, 7 mod 10	3, 4, 10 mod 11	6, 8, 9 mod 14
2 mod 4	7, 8 mod 9	0, 1, 4 mod 10	5, 7, 8 mod 10	3, 6, 10 mod 11	7, 8, 12 mod 14
2 mod 6	4, 7 mod 11	0, 1, 6 mod 10	0, 1, 5 mod 11	1, 7, 10 mod 12	8, 9, 12 mod 14
2 mod 9	5, 6 mod 11	0, 1, 8 mod 10	0, 1, 8 mod 11	1, 8, 11 mod 12	1, 5, 7 mod 15
0, 3 mod 4	6, 8 mod 11	0, 2, 4 mod 10	0, 1, 9 mod 11	2, 4, 11 mod 12	1, 5, 11 mod 15
0, 1 mod 5	6, 9 mod 11	0, 2, 6 mod 10	0, 2, 5 mod 11	4, 7, 10 mod 12	1, 5, 13 mod 15
0, 2 mod 5	1, 5 mod 12	0, 2, 7 mod 10	0, 2, 8 mod 11	1, 3, 4 mod 13	1, 5, 14 mod 15
1, 3 mod 5	2, 5 mod 12	0, 2, 8 mod 10	0, 4, 5 mod 11	1, 4, 6 mod 13	1, 7, 8 mod 15
2, 3 mod 5	2, 8 mod 12	0, 4, 7 mod 10	0, 4, 8 mod 11	1, 8, 11 mod 13	1, 8, 13 mod 15
1, 4 mod 6	2, 10 mod 12	0, 6, 7 mod 10	0, 4, 9 mod 11	2, 3, 7 mod 13	1, 8, 14 mod 15
1, 5 mod 6	4, 5 mod 12	0, 7, 8 mod 10	1, 2, 7 mod 11	2, 6, 7 mod 13	1, 10, 11 mod 15
4, 5 mod 6	5, 8 mod 12	1, 3, 4 mod 10	1, 3, 5 mod 11	3, 4, 9 mod 13	1, 10, 13 mod 15
2, 3 mod 7	7, 8 mod 12	1, 3, 6 mod 10	1, 3, 8 mod 11	3, 4, 10 mod 13	2, 5, 7 mod 15
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3, 4 mod 7	1, 8 mod 18	1, 4, 5 mod 10	1, 3, 10 mod 11	3, 10, 11 mod 13	2, 5, 13 mod 15
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2, 6 mod 8	10, 11 mod 18	2, 3, 8 mod 10	2, 3, 8 mod 11	7, 8, 9 mod 13	2, 10, 11 mod 15
3, 4 mod 8	5, 11 mod 21	2, 4, 5 mod 10	2, 3, 9 mod 11	8, 9, 11 mod 13	2, 10, 13 mod 15
3, 5 mod 8	0, 1, 3 mod 7	2, 5, 6 mod 10	2, 3, 10 mod 11	8, 10, 11 mod 13	2, 10, 14 mod 15
4, 6 mod 8	0, 1, 5 mod 7	2, 5, 7 mod 10	2, 5, 7 mod 11	3, 4, 10 mod 14	4, 5, 11 mod 15
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   We're still working on a general answer, but we know that many (such as 2 mod 9) occur in all of them!

Question 5.

Which deeper structure theorems about T-orbits can be used to improve on these results?

Background on percentage of 1's in a T-orbit

- **Theorem.** (Eliahou, 1993.) If a T-cycle of positive integers of length n contains r odd positive integers (and $n - r$ even positive integers), and has minimal element m and maximal element M, then

$$\frac{\ln(2)}{\ln\left(3 + \frac{1}{m}\right)} \leq \frac{r}{n} \leq \frac{\ln(2)}{\ln\left(3 + \frac{1}{M}\right)}$$

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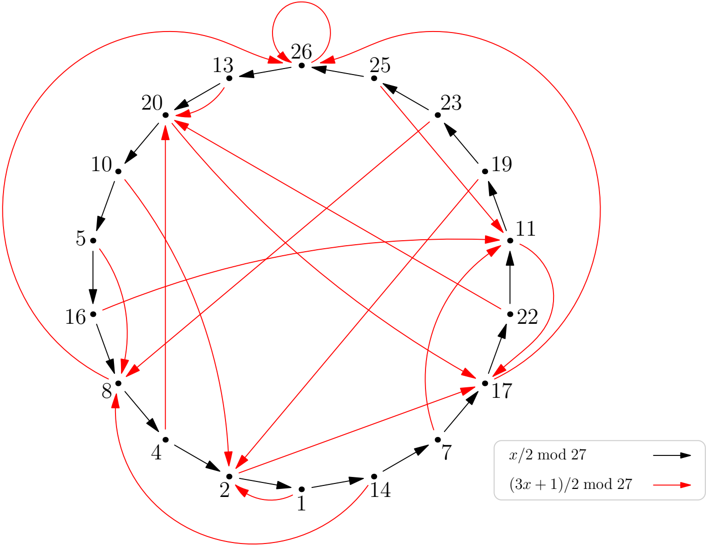
- **Theorem.** (Lagarias, 1985.) Similarly, the percentage of 1's in any divergent orbit is *at least* $\ln(2)/\ln(3) \approx .6309$.
- With these facts, we can show 20 mod 27 is strongly sufficient in the forward direction.

Looking mod 27



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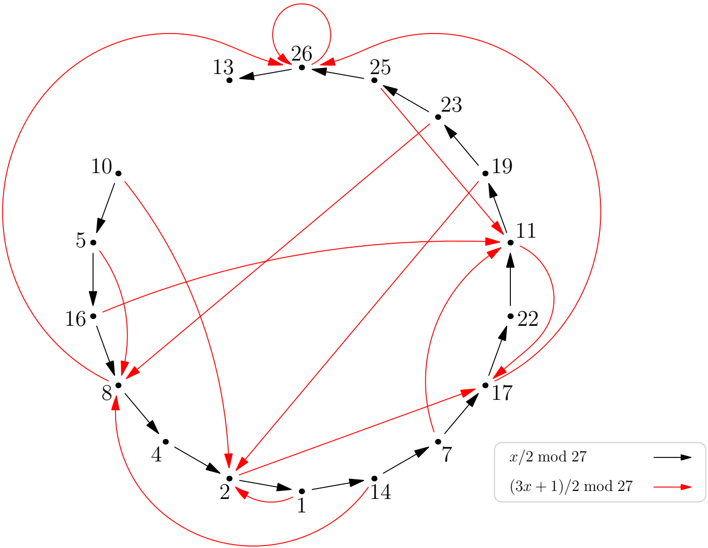

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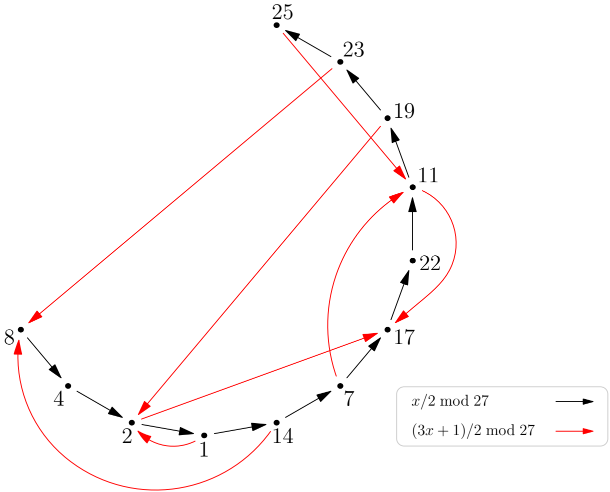

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- The percentage of 1's in any divergent orbit or nontrivial cycle is at least 63%. This can be used to obtain more strongly sufficient sets.

Background on T as a 2-adic dynamical system

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- **Theorem.** (Bernstein, Lagarias.) Inverse parity vector function

$$\Phi : \mathbb{Z}_2 \to \mathbb{Z}_2$$

is well defined, and

$$T = \Phi \circ \sigma \circ \Phi^{-1}.$$

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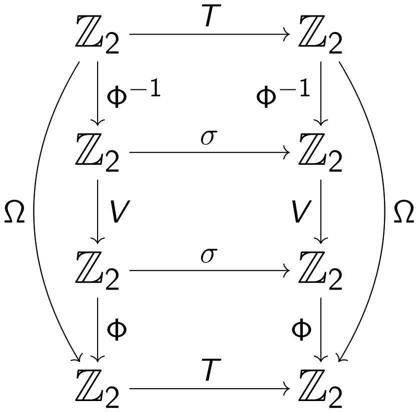
- $V(100100100\ldots) = 011011011\ldots$.
- In 2004, K. G. Monks and J. Yasinski used V to construct the unique nontrivial autoconjugacy of T:

$$\Omega := \Phi \circ V \circ \Phi^{-1}.$$

The autoconjugacy Ω



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- We say that, mod 8, $\Omega(3) = 4$.

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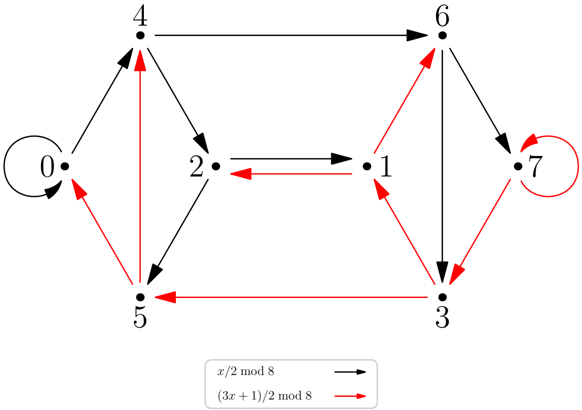
Idea of proof: if we replace each label a with $\Omega(a) \bmod 2^n$, we get the color dual of Γ_{2^n}.

Example: Γ_8



Hedlund's other endomorphisms

- **Discrete derivative map:** $D : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $D(a_0 a_1 a_2 \ldots) = d_0 d_1 d_2 \ldots$ where $d_i = |a_i - a_{i+1}|$ for all i.

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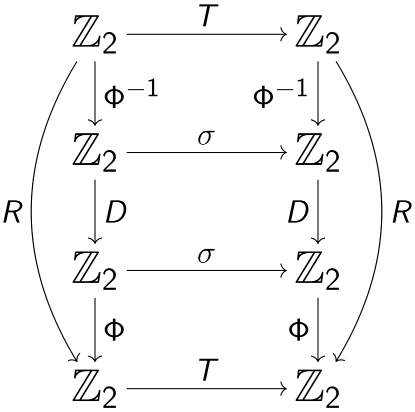
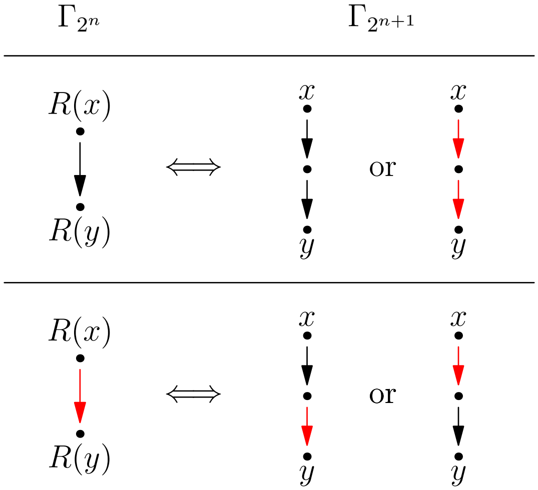
- (M., 2009.) R is a two-to-one map, and $R(\Omega(x)) = R(x)$ for all x.
- Can use R to "fold" $\Gamma_{2^{n+1}}$ onto Γ_{2^n} by identifying Ω-pairs.

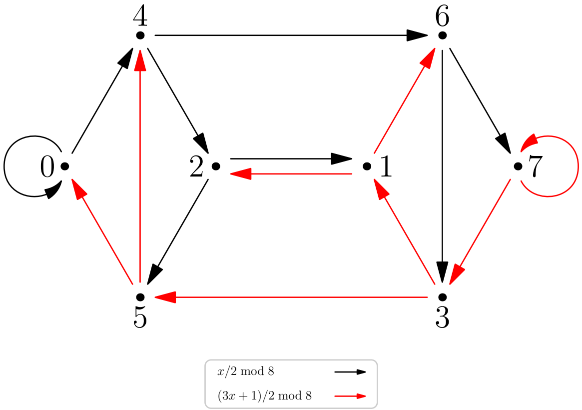
The endomorphism R



Folding $\Gamma_{2^{n+1}}$ onto Γ_{2^n}



Folding Γ_8 onto Γ_4



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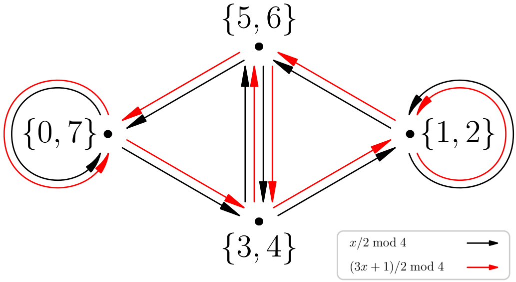

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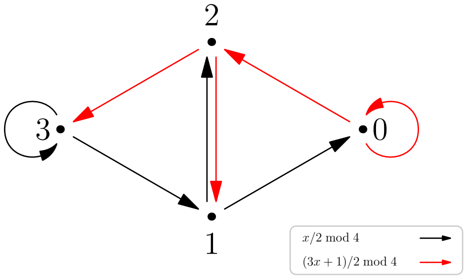

Question 5.

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- *The structure of T as a 2-adic dynamical system can be used to obtain properties of the graphs Γ_{2^n}.*

Future work

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- Are there other graph-theoretic techniques that would be useful?
- Can we find an irrational infinite back-tracing parity vector explicitly, say using algebraic properties?

Acknowledgements

The authors would like to thank Gina Monks for her support throughout this research project.