Characterization of queer supercrystals

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On joint work with Graham Hawkes, Wencin Poh, and Anne Schilling

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May 30, 2019

Why 'Crystals'?

- Crystals arise at cold temperatures!
- Kashiwara: 'crystal bases' of representations of quantum groups $U_q(\mathfrak{g})$ in the limit $q \to 0$ (q is temperature).
- Rigid combinatorial structures with applications to symmetric function theory, representation theory, geometry...

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Talk outline:

- Part 1: Type A crystals (for Lie algebra $\mathfrak{g} = \mathrm{sl}_n$)
- Part 2: Queer supercrystals (for quantum queer Lie superalgebra $q(n)$)

Lie algebras: Notation and Background

Notation	Example/Description
Lie algebra $\mathfrak{g}$	$\mathfrak{sl}_n$ (trace-0 $n \times n$ matrices)
Lie bracket $[,]$	$[x, y] = xy - yx$
Classical types: A_n, B_n, C_n, D_n	
Weight lattice Λ	
Simple roots α_i, $i \in I$	
Generators e_i, f_i, h_i	
Univ. envel. alg. $U(\mathfrak{g})$	
Quantized UEA $U_q(\mathfrak{g})$	

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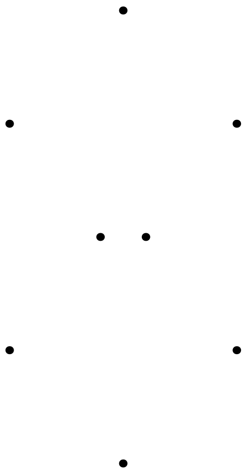
Quantized UEA $U_q(\mathfrak{g})$	$\lim_{q \to 1} U_q(\mathfrak{g}) = U(\mathfrak{g})$ $q \to 0$: crystal bases for reps

Lie algebra crystals

(Ex. $\mathfrak{g} = \mathfrak{sl}_3$)

- **Ground set** $\mathcal{B}$ ("base")



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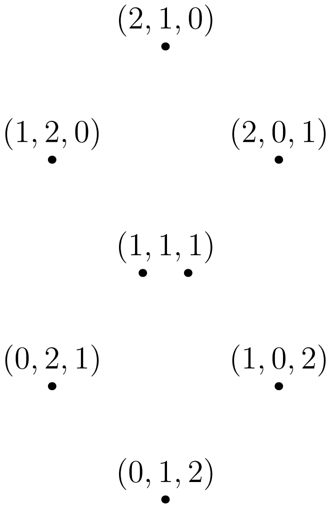

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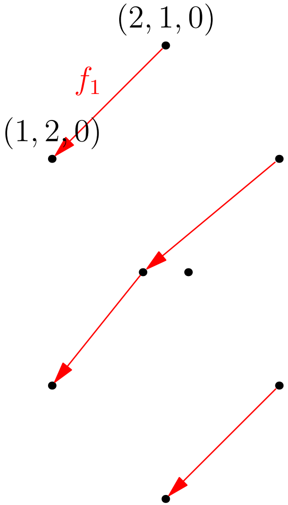

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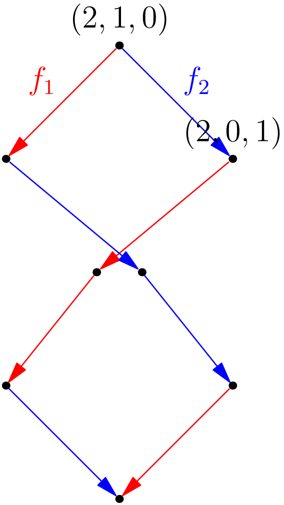

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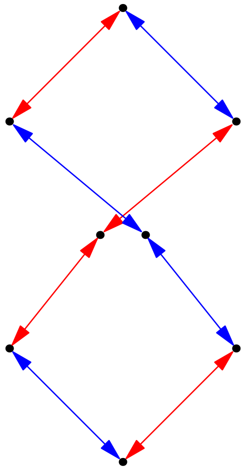

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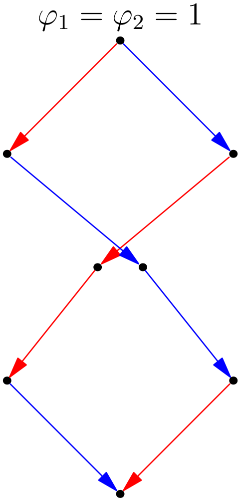
- **Lengths** $\varphi_i, \varepsilon_i : \mathcal{B} \to \mathbb{Z}$, usually:

$$\varphi_i(x) = \max\{k : f_i^k(x) \neq 0\}$$

$$\varepsilon_i(x) = \max\{k : e_i^k(x) \neq 0\}$$



Stembridge crystals

- **Stembridge**: 'Local axioms' determine which crystals correspond to $U_q(\mathfrak{g})$-representations (for simply-laced types).
  - **Lengths Axiom:**
  - **Non-adjacent operators:**
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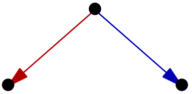	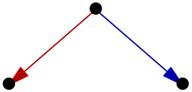

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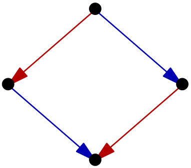	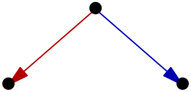

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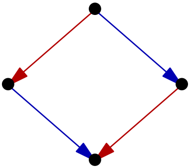	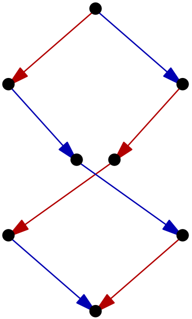

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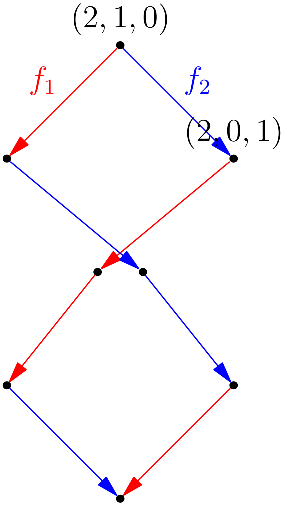
- Unique **highest weight** elements (killed by all e_i operators)



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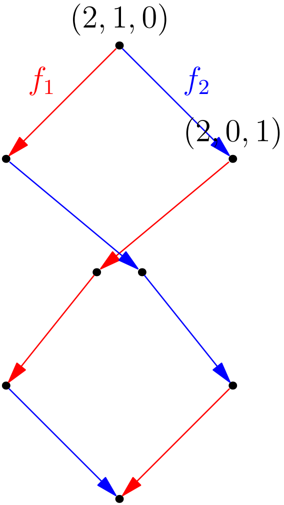

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$$\sum_{b \in \mathcal{B}} x^{\mathrm{wt}(b)}$$

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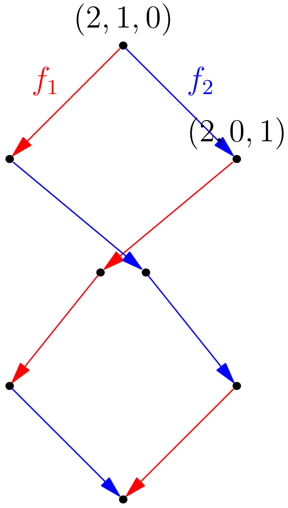

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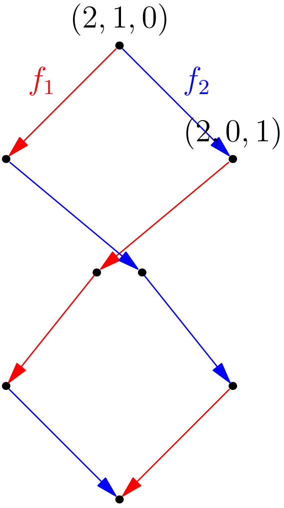
- Can recover Littlewood-Richardson rule:

$$s_\lambda s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$$

  via crystal **tensor products**



Tensor products of crystals

Tensor product $\mathcal{B} \otimes \mathcal{C}$ is the crystal having:

- Ground set $\mathcal{B} \times \mathcal{C}$
- Weight function $\mathrm{wt}(x \otimes y) = \mathrm{wt}(x) + \mathrm{wt}(y)$
- Operators

$$e_i(x \otimes y) = \begin{cases} e_i(x) \otimes y & \varphi_i(y) < \varepsilon_i(x) \\ x \otimes e_i(y) & \varphi_i(y) \geqslant \varepsilon_i(x) \end{cases}$$

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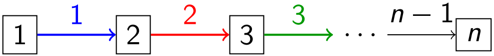
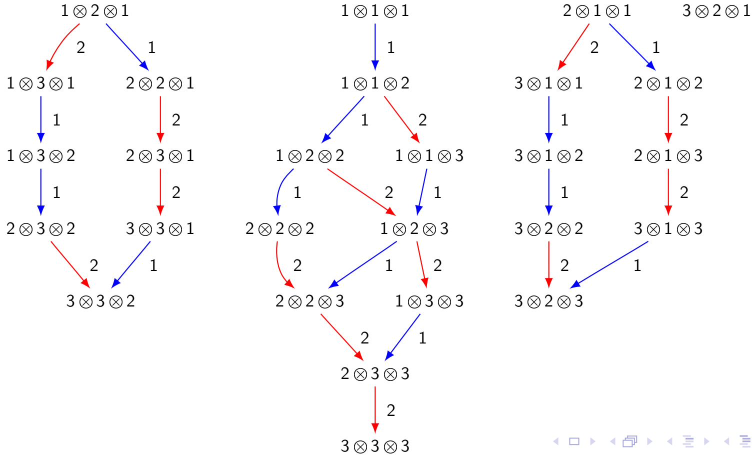
Standard crystal and tensor products

Standard crystal $\mathcal{B}_0$ for $\mathfrak{sl}_n$:



Components of **crystal of words** $\mathcal{B}_0^{\otimes 3} = \mathcal{B}_0 \otimes \mathcal{B}_0 \otimes \mathcal{B}_0$ for $\mathfrak{sl}_3$:



Part 2: Lie superalgebras and $q(n)$

- **Lie superalgebra**: $\mathbb{Z}_2$-graded algebra $\mathfrak{g}_0 \oplus \mathfrak{g}_1$ with 'super' Lie bracket $[,]$. Example:

$$[x, y] = \begin{cases} xy - yx & x \in \mathfrak{g}_0 \text{ or } y \in \mathfrak{g}_0 \\ xy + yx & x, y \in \mathfrak{g}_1 \end{cases}$$

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- 'Classical' Lie superalgebras (simple, $\mathfrak{g}_1$ is reducible $\mathfrak{g}_0$-rep):
  - Main series: $A(m, n)$, $B(m, n)$, $C(n)$, $D(m, n)$
  - Deformations: $D(2, 1; \alpha)$
  - Exceptional: $G(3)$, $F(4)$
  - Strange: $P(n)$, $Q(n)$ (also analog of type A Lie algebra)

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- Type $Q(n)$: queer Lie superalgebra $q(n) \simeq \mathfrak{sl}_n \oplus \mathfrak{sl}_n$, generators e_i, f_i, h_i for $q(n)_0$, plus generators f_{-1}, e_{-1}, h_{-1} for $q(n)_1$

$q(n)$ crystals

- Grantcharov, Jung, Kang, Kashiwara, Kim '10: Crystal bases for $U_q(q(n))$ representations ('quantum queer supercrystals')

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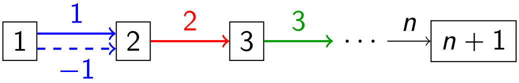
- Standard queer crystal $\mathcal{B}_0$:



- **Tensor products:** Type A rules for positive arrows, and:

$$f_{-1}(b \otimes c) = \begin{cases} b \otimes f_{-1}(c) & \text{if } \mathrm{wt}(b)_1 = \mathrm{wt}(b)_2 = 0 \\ f_{-1}(b) \otimes c & \text{otherwise} \end{cases}$$

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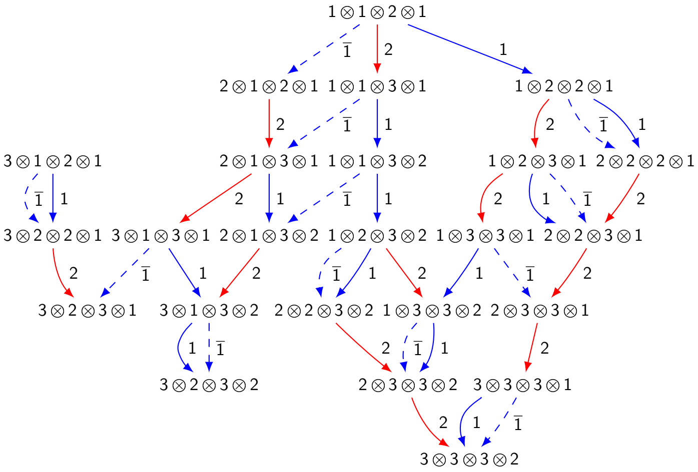
- **Characters:** Schur P-functions
- **QUESTION:** Stembridge-like local characterization of queer crystal graphs?

$q(n)$ crystals

One connected component of $\mathcal{B}_0^{\otimes 4}$ for $\mathfrak{q}(3)$:

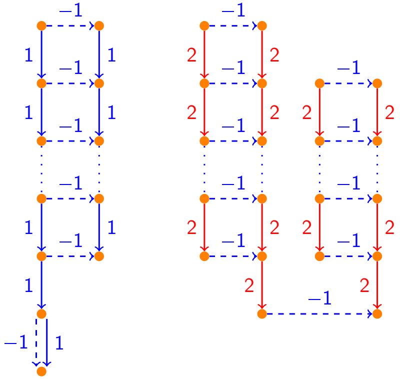


Notice 'fake highest weight' element $3 \otimes 1 \otimes 2 \otimes 1$.

Restricting to $-1, 1$ or $-1, 2$ arrows

Conjecture (Assaf, Oguz '18)

In addition to the Stembridge axioms for the positive arrows and the assumption that -1 arrows commute with all i-arrows for $i \geqslant 3$, the relations below uniquely characterize queer crystals.



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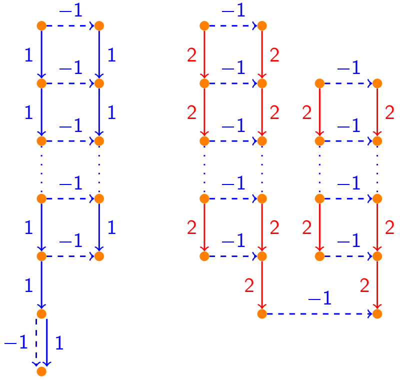


(GHPS) A counterexample exists!

Further axioms

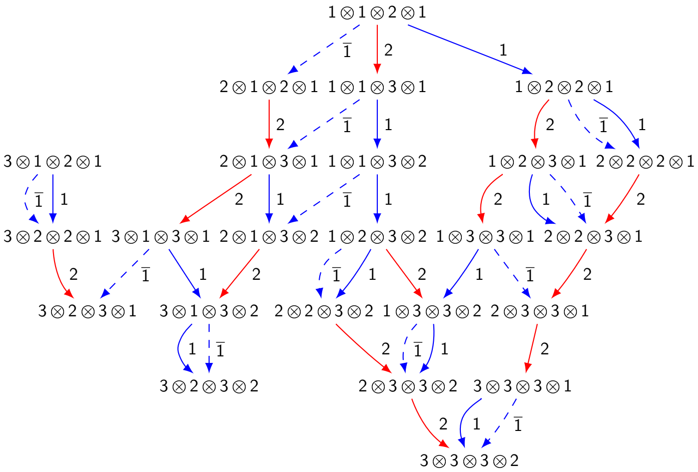
Can add extra axioms to entirely characterize $q(n)$ crystals.
Require:

Definition

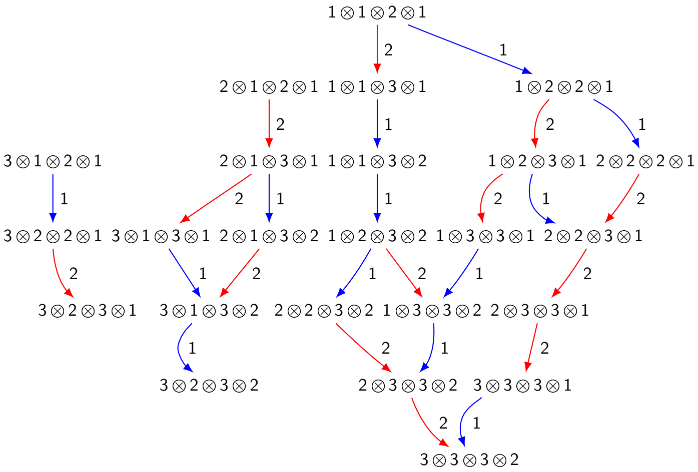
Type A component graph $G(\mathcal{C})$:

- Delete -1 arrows; remaining arrows are 'type A'
- Replace each type A component with a single vertex labeled by highest weight; edge between them if -1 arrow between them.

Component graph



Component graph



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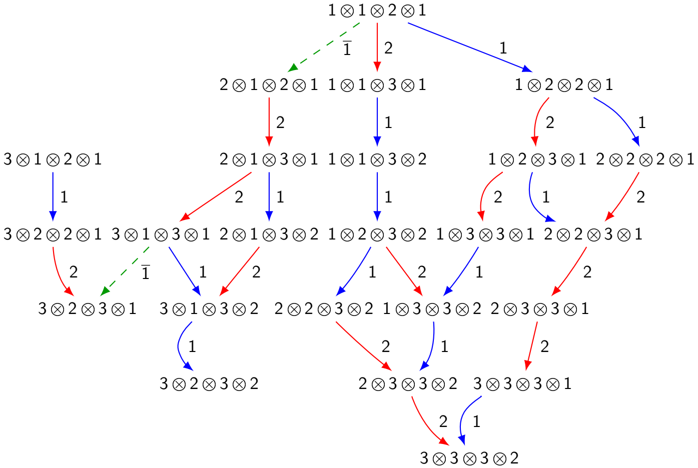

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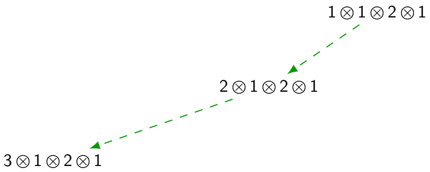
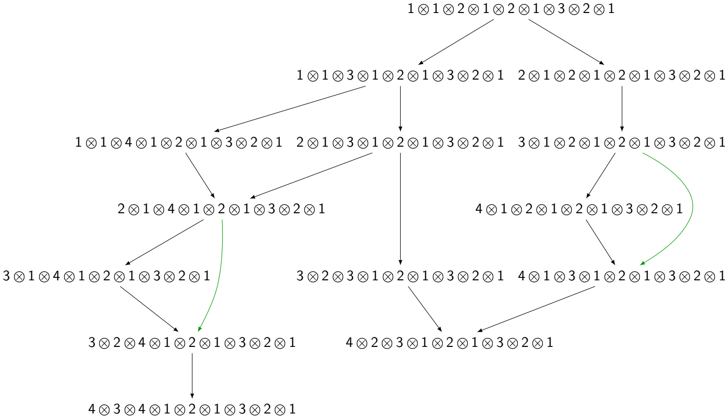


Another component graph



- Gives expansion of P-schur function P_λ in terms of Schur functions s_μ.

Combinatorial description of $G(\mathcal{C})$

- Define

$$f_{-i} := s_{w_i}^{-1} f_{-1} s_{w_i} \qquad \text{and} \qquad e_{-i} := s_{w_i}^{-1} e_{-1} s_{w_i}$$

  where $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$ and s_i is reflection along i-string
- Adding in $-i$ arrows removes fake highest weights [GJKKK]

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Define $f_{(-i,h)} := f_{-i} f_{i+1} f_{i+2} \cdots f_{h-1}$.

Proposition (GHPS)

Minimal set of edges to connect $G(\mathcal{C})$: starting at highest weight, apply $f_{(-i,h)}$ to each vertex v for some i and $h > i$ minimal such that $f_{(-i,h)}(v)$ is defined.

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Proposition (GHPS)

Edge $C_1 \to C_2$ is in $G(\mathcal{C})$ iff

$$e_{-i} u_2 \in C_1$$

for some i, where u_2 is the highest weight element of C_2.

Combinatorial description of $G(\mathcal{C})$ (continued)

Theorem (GHPS)

There are explicit combinatorial algorithms for computing f_{-i} and e_{-i} on type A highest weight words.

Algorithm for f_{-i}:

- b: highest weight word. Ex: $b = 545423321211$

Combinatorial description of $G(\mathcal{C})$ (continued)

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- Find leftmost $i, i-1, \ldots, 1$. Ex: $b = \overline{5}\overline{4}542\overline{3}3\overline{2}\overline{1}211$

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Combinatorial description of $G(\mathcal{C})$ (continued)

Theorem (GHPS)

There are explicit combinatorial algorithms for computing f_{-i} *and* e_{-i} *on type A highest weight words.*

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Similar algorithms for e_{-i} and determining if f_{-i}, e_{-i} defined.

Main Theorem: Characterization

Theorem (GHPS)

Let $\mathcal{C}$ be a connected component of a generic abstract queer crystal such that:

1. *$\mathcal{C}$ satisfies the local axioms of Stembridge, Assaf and Oguz*
2. *The* **component graph** *$G(\mathcal{C})$ matches $G(\mathcal{D})$ for some connected component $\mathcal{D}$ of $\mathcal{B}^{\otimes \ell}$*
3. *$\mathcal{C}$ satisfies three extra* **connectivity axioms***. (Put back all -1 arrows.)*

Then $\mathcal{C}$ is a queer supercrystal and $\mathcal{C} \cong \mathcal{D}$.

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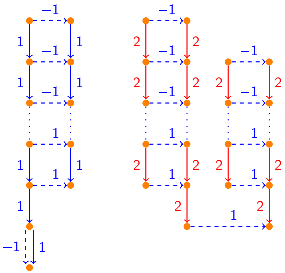
Thank you!

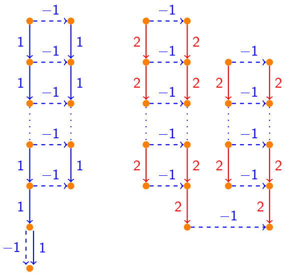
Connectivity axioms: Almost lowest weight elements



Almost lowest weight elements:

$$\varphi_1(b) = 2 \quad \text{and} \quad \varphi_i(b) = 0 \quad \text{for all } i \in I_0 \backslash \{1\}$$

Connectivity axioms: Almost lowest weight elements



Almost lowest weight elements:

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Lemma

Almost lowest weight elements are $g_{j,k} := (e_1 \cdots e_j)(e_1 \cdots e_k)v$, *where* v *is lowest weight and* $1 \leqslant j \leqslant k \leqslant n$.

Connectivity axioms

C0. $\varphi_{-1}(g_{j,k}) = 0$ implies that $\varphi_{-1}(e_1 \cdots e_k v) = 0$.

C1. If $G(\mathcal{C})$ contains edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$.
Then for all $h < j \leqslant k$,

$$f_{-1} g_{j,k} = (e_2 \cdots e_j)(e_1 \cdots e_h) v'$$

where v' is I_0-lowest weight with $\uparrow v' = u'$.

C2. (a) $G(\mathcal{C})$ contains edge $u \to u'$ such that $\mathrm{wt}(u')$ is obtained from $\mathrm{wt}(u)$ by moving a box from row $n+1-k$ to row $n+1-h$ with $h < k$ or

(b) no such edge exists in $G(\mathcal{C})$

Then for all $1 \leqslant j \leqslant h$ in case (a) and all $1 \leqslant j \leqslant k$ in case (b)

$$f_{-1} g_{j,k} = (e_2 \cdots e_k)(e_1 \cdots e_j) v.$$