Monodromy and K-theory of Schubert curves via generalized jeu de taquin

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- **Rephrasing:** How many planes through $(0, 0)$ in $\mathbb{C}^4$ intersect four such planes in a line? Work in $\mathrm{Gr}(4, 2)$.
- **Schubert variety:** For the complete flag

$$\mathcal{F} : \quad \varnothing = F_0 \subset F_1 \subset F_2 \subset \cdots \subset F_n = \mathbb{C}^n,$$

$$\Omega_\lambda(\mathcal{F}) = \{V \in \mathrm{Gr}(n, k) : \dim V \cap F_{n-k+i-\lambda_i} \geqslant i\}$$

  where λ fits inside a $k \times (n - k)$ rectangle $\ydiagram{3,3}$.

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- **Littlewood-Richardson rule:** (Generalized.) Let $\lambda_1, \dots, \lambda_r \subset \boxplus$ be partitions with $\sum |\lambda_i| = m$. Then in $H^*(\mathrm{Gr}(n,k))$, the classes $[\Omega_\lambda]$ form a basis and

$$[\Omega_{\lambda_1}] \cdot \dots \cdot [\Omega_{\lambda_r}] = \sum_{|\nu|=m} c^{\nu}_{\lambda_1,\dots,\lambda_r} \cdot [\Omega_\nu].$$

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- Example: The coefficient $c^{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}_{\square,\square,\square,\square}$ is 2, and

$$[\Omega_\square(\mathcal{F}^{(1)}) \cap \Omega_\square(\mathcal{F}^{(2)}) \cap \Omega_\square(\mathcal{F}^{(3)}) \cap \Omega_\square(\mathcal{F}^{(4)})] = c^{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}_{\square,\square,\square,\square}[\Omega_{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}].$$

Note that $\Omega_{\begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}}(\mathcal{F})$ is a single point for any flag $\mathcal{F}$.

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- The coefficient $c^{\nu}_{\lambda_1, \ldots, \lambda_r}$ counts the number of ways of filling ν with a chain of skew **Littlewood-Richardson tableaux** with contents $\lambda_1, \ldots, \lambda_r$.

Essential terminology

- **Littlewood-Richardson tableau** of shape ν/μ: *Semistandard* tableau whose *reading word* is *ballot* (a.k.a. *Yamanouchi*).

$$
\begin{array}{cccccc}
 & & & 1 & 1 & 1 \\
 & \mu & & 2 & 2 & \\
 & 1 & 2 & 3 & 3 & \\
2 & 3 & 4 & 4 & & \\
3 & 5 & & & &
\end{array}
\qquad
\begin{aligned}
\mu &= (3,3,1) \\
\nu &= (6,5,5,4,1)
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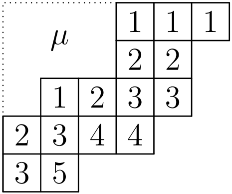

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- **Content:** The sequence $(m_1, m_2, \ldots)$ where m_i is the number of i's in the tableau. Here the content is $(4,4,4,2,1)$.

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- Special Schubert curves: flags $\mathcal{F}_{t_i}$ are maximally tangent at real points of the **rational normal curve** in $\mathbb{P}^{n-1}$:

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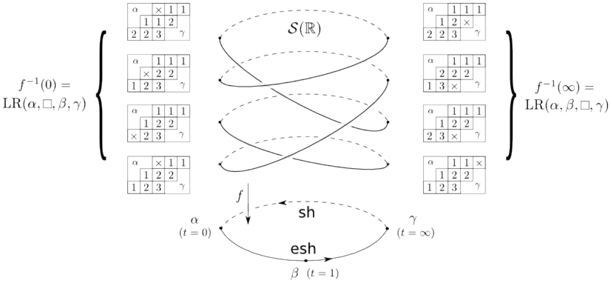
- We restrict to the case of three real points, three partitions α, β, γ with $|\alpha| + |\beta| + |\gamma| = k(n-k) - 1$; this is sufficient generality to demonstrate our results. Define

$$\boxed{S = S(\alpha,\beta,\gamma) = \Omega_\alpha(\mathcal{F}_0) \cap \Omega_\beta(\mathcal{F}_1) \cap \Omega_\gamma(\mathcal{F}_\infty)}$$

The Setup: Real geometry of S

- $S(\mathbb{R})$ - real points of S



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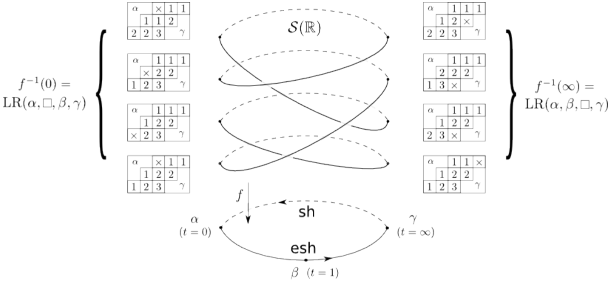

Theorem. (Levinson, based on work of Speyer, Mukhin-Tarasov-Varchenko, Eisenbud-Harris, and others.) There is a map $S \to \mathbb{P}^1$ that makes $S(\mathbb{R})$ a smooth covering of the circle $\mathbb{RP}^1$, with finite fibers of size

$$c^{\ydiagram{3,2}}_{\alpha,\beta,\gamma} = c^{\ydiagram{3,3}}_{\alpha,\beta,\gamma,\square} = c^{\ydiagram{3,3}}_{\alpha,\square,\beta,\gamma}.$$

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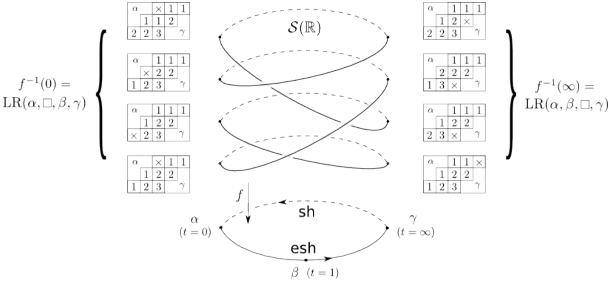

- (Fiber over 0) $\leftrightarrow \mathrm{LR}(\alpha, \square, \beta, \gamma)$, the set of tableaux of shape γ^c/α with one **inner** corner marked as the "special box" and the rest a Littlewood-Richardson tableau of content β.

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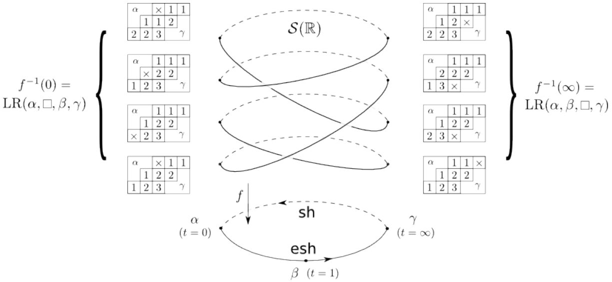

- The arcs of $S(\mathbb{R})$ covering $\mathbb{R}_-$ and $\mathbb{R}_+$ respectively induce the *shuffling* and *evacuation-shuffling* bijections sh and esh:

$$\mathrm{LR}(\alpha, \square, \beta, \gamma) \underset{\mathrm{sh}}{\overset{\mathrm{esh}}{\rightleftarrows}} \mathrm{LR}(\alpha, \beta, \square, \gamma)$$

- **Monodromy operator:** $\omega = \mathrm{sh} \circ \mathrm{esh}$

Shuffling

- **Shuffling**, or **JDT**: Do an outer jeu de taquin slide with the $\boxtimes$ as the empty square to get an element of $\mathrm{LR}(\alpha, \square, \beta, \gamma)$.

α			1	1	1
		1	2	2	2
	2	3	3		
1	3	4	4	γ	
3	4	5	$\times$		

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Evacuation-shuffling

- Conjugation of shuffling by JDT rectification.
- **Rectification:** Treat $\times$ as 0.

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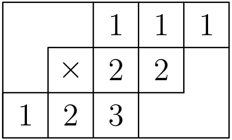

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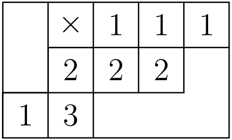

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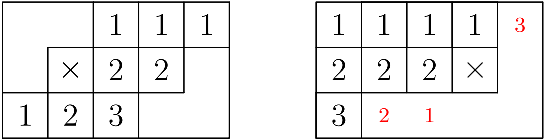

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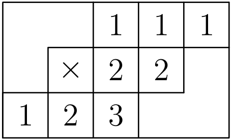

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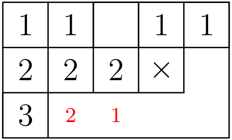

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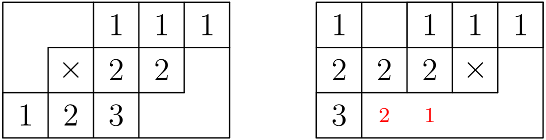

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T $\qquad$ $\mathrm{esh}(T)$

- **Shuffle again** to compute $\omega = \mathrm{sh} \circ \mathrm{esh}$:

$$\omega T: \quad \begin{array}{ccccc} & & \times & 1 & 1 \\ & 1 & 1 & 2 & \\ 2 & 2 & 3 & & \end{array}$$

Background on K-theory

- K-theory ring $K(\mathrm{Gr}(n,k))$ has additive basis $[\mathcal{O}_\lambda]$.

$$\begin{aligned} [\mathcal{O}_S] &= [\mathcal{O}_\alpha]\cdot[\mathcal{O}_\beta]\cdot[\mathcal{O}_\gamma] \\ &= \sum_{|\nu|\geqslant|\alpha|+|\beta|+|\gamma|} (-1)^{|\nu|-|\alpha|-|\beta|-|\gamma|} k^{\nu}_{\alpha,\beta,\gamma}[\mathcal{O}_\nu] \end{aligned}$$

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$$
\begin{aligned}
\chi(\mathcal{O}_S) &= c^{\ydiagram{3,2}}_{\alpha,\beta,\gamma} - k^{\ydiagram{3,3}}_{\alpha,\beta,\gamma} \\
&= |\mathrm{LR}(\alpha,\square,\beta,\gamma)| - k^{\ydiagram{3,3}}_{\alpha,\beta,\gamma}
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Computing $k_{\alpha,\beta,\gamma}^{\ydiagram{3,3}}$

- Pechenik and Yong introduced **genomic tableaux**.

Computing $k^{\ydiagram{3,3}}_{\alpha,\beta,\gamma}$

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- The data $(T, \{\boxtimes_1, \boxtimes_2\})$ corresponds to a genomic tableau if
  - (i) The squares are non-adjacent and contain the same entry i,
  - (ii) There are no i's between $\boxtimes_1$ and $\boxtimes_2$ in the reading word of T,
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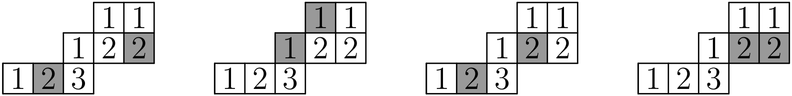
- *Which of the following are genomic tableaux?*



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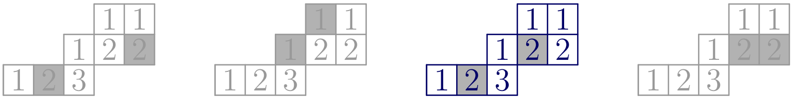

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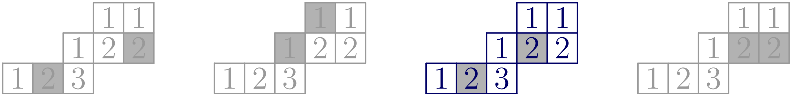

- ***K*-theoretic content:** $\beta = (4, 2, 1)$
- Let $K(\gamma^c/\alpha; \beta)$ be the set of genomic tableaux of shape γ^c/α and K-theoretic content β. Then

$$K := k^{\ydiagram{3,3}}_{\alpha,\beta,\gamma} = |K(\gamma^c/\alpha; \beta)|.$$

Geometric connections to K-theory

- (Levinson.) Let $\eta(S)$ be the number of connected components of $S(\mathbb{R})$, so $\eta(S) = |\mathrm{orbits}(\omega)|$. Then

$$\begin{aligned} \eta(S) &\geqslant \chi(\mathcal{O}_S) && (1) \\ \eta(S) &\equiv \chi(\mathcal{O}_S) \pmod 2 && (2) \end{aligned}$$

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$$K \geqslant |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{orbits}(\omega)| \tag{3}$$

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Is there a combinatorial explanation?

Local rule for evacuation-shuffling

- Recall: esh consists of rectifying, shuffling, and un-rectifying.
- (G., Levinson.) Local rule, without rectifying: Start at $i = 1$.

						1	1	1
			$\times$	1	1	2	2	
			1	2	2	3		
			3	3	4			
		4	4					
2	3	5						

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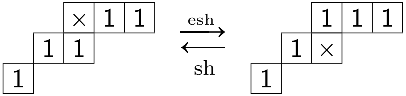
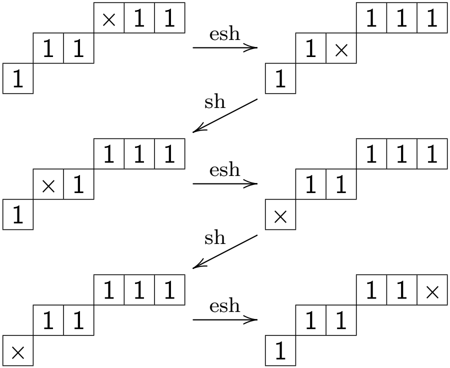
			2	2	2	3		
			3	3	$\times$			
		4	4					
3	4	5						

Proof of local rule

- First prove it directly for horizontal strips, $\beta = (m)$.
- "Pieri Case:" Moves downwards to next row cyclically.





Proof of local rule

- Proof of Pieri Case:

$$
\begin{array}{cccccc}
 & & & \times & 1 & 1 \\
 & 1 & 1 & & & \\
1 & & & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
 & & & \times & 1 & 1 \\
 & 1 & 1 & & & \\
1 & & & & &
\end{array}
$$

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\begin{array}{cccccc}
 & & & \times & 1 & 1 \\
 & 1 & 1 & & & \\
1 & & & & &
\end{array}
\qquad\qquad
\begin{array}{cccccc}
 & & \times & 1 & 1 & {\color{red}1} \\
 & 1 & 1 & & & \\
1 & & & & &
\end{array}
$$

Proof of local rule

- Proof of Pieri Case:

$$\begin{array}{|c|c|c|c|c|c|}
\cline{4-6} \multicolumn{3}{c|}{} & \times & 1 & 1 \\
\cline{2-6} \multicolumn{1}{c|}{} & 1 & 1 & \multicolumn{3}{c}{} \\
\cline{1-3} 1 & \multicolumn{5}{c}{} \\
\cline{1-1}
\end{array}
\qquad\qquad
\begin{array}{|c|c|c|c|c|l}
\cline{2-5} \multicolumn{1}{c|}{} & \times & 1 & 1 & 1 & \color{red}{1} \\
\cline{2-5} \multicolumn{1}{c|}{} & 1 & \multicolumn{4}{l}{\color{red}{2}} \\
\cline{1-2} 1 & \multicolumn{5}{c}{} \\
\cline{1-1}
\end{array}$$

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Proof of local rule

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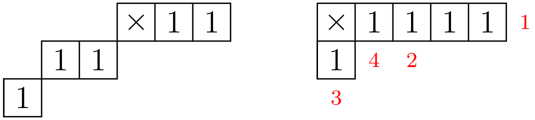

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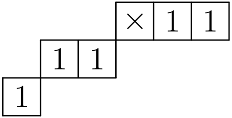

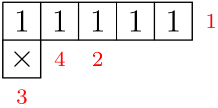

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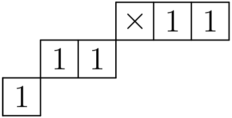

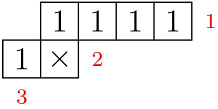

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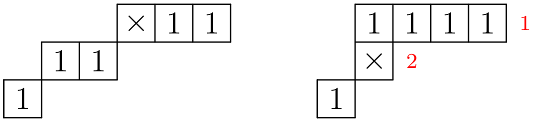

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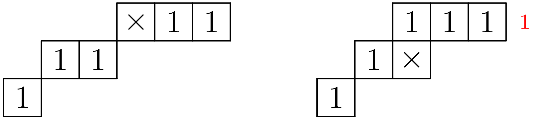

Proof of local rule

- Proof of Pieri Case:

			$\times$	1	1
	1	1			
1					

			1	1	1
	1	$\times$			
1					

Proof of local rule

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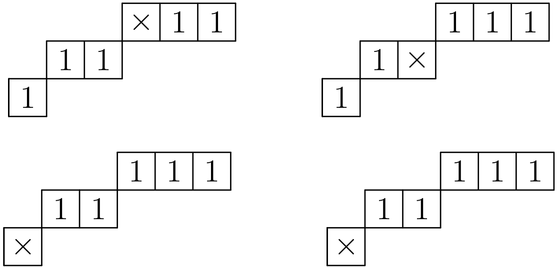

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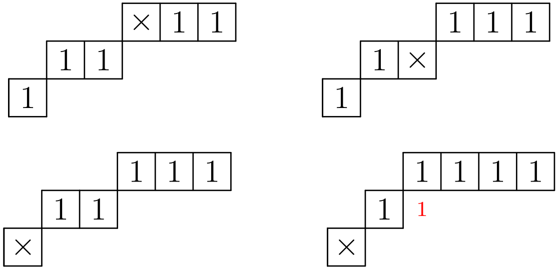

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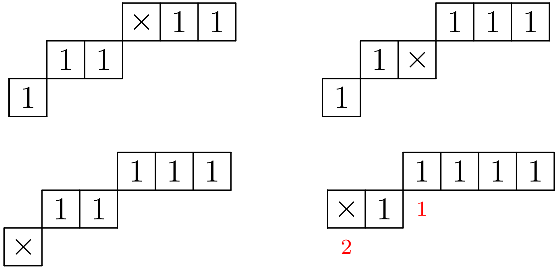

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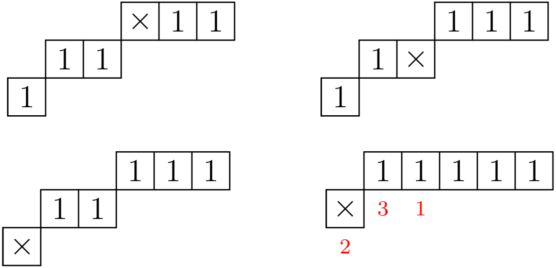

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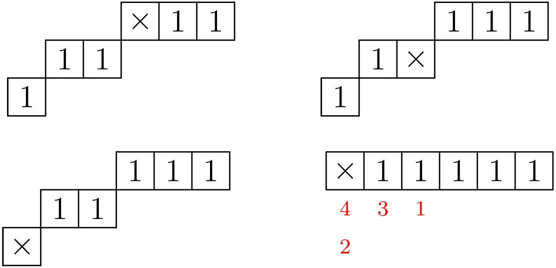

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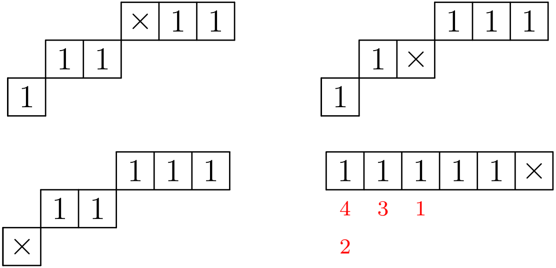

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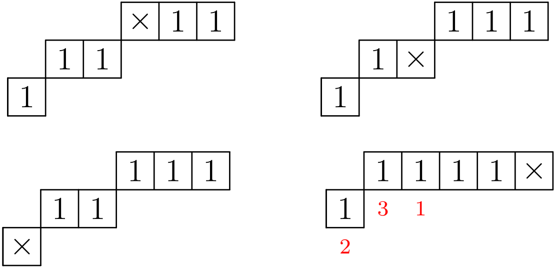

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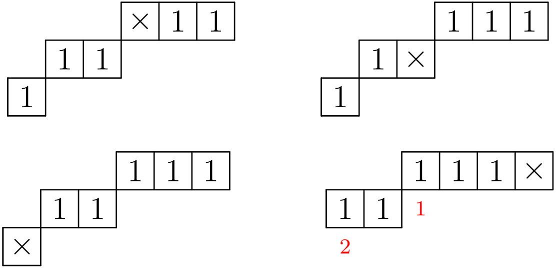

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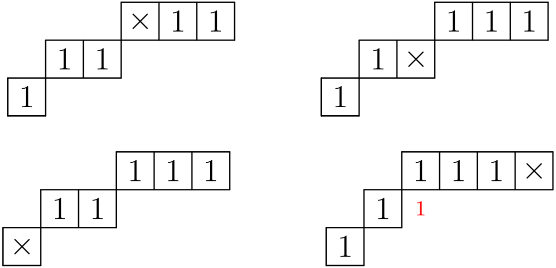

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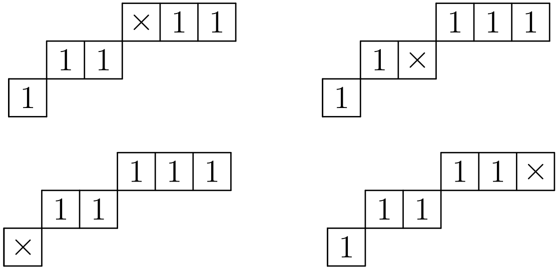

Proof of local rule

- General case: "Factor" the tableau into horizontal and vertical strips.

$\times$	1	1	1	1	1
1	2	2	2	2	
2	3	3	3	3	
4	4	4			
5					

						1	1	1
			$\times$	1	1	2	2	
			1	2	2	3		
			3	3	4			
		4	4					
2	3	5						

- Switch the $\boxtimes$ with the first element prior to it in a horizontal strip (as in Pieri case) or with the next element after it in a vertical strip (conjugate of Pieri case).

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$\times$	2	2	2	2	
2	3	3	3	3	
4	4	4			
5					

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$$
\begin{array}{|c|c|c|c|c|c|}
\hline
1 & 1 & 1 & 1 & 1 & 1 \\ \hline
2 & 2 & 2 & 2 & 2 \\ \cline{1-5}
\times & 3 & 3 & 3 & 3 \\ \cline{1-5}
4 & 4 & 4 \\ \cline{1-3}
5 \\ \cline{1-1}
\end{array}
\qquad
\begin{array}{ccccccccc}
 & & & & & & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 & 2 & 2 & \\
 & & & 2 & 2 & 2 & 3 & & \\
 & & & 3 & 3 & 4 & & & \\
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\times & 3 & 5 & & & & & &
\end{array}
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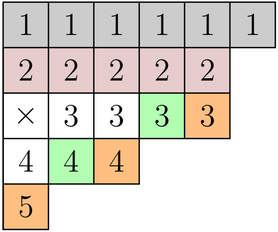

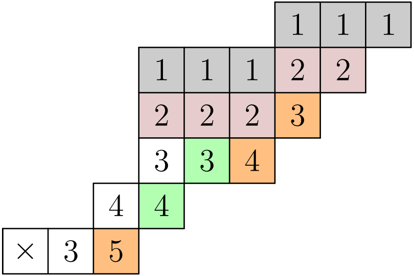

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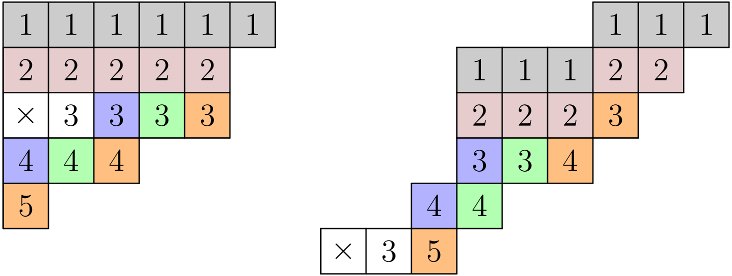

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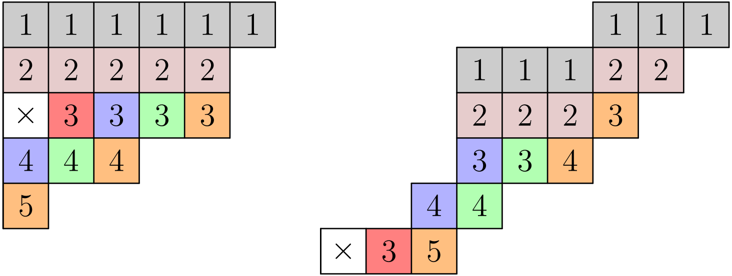

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3	×	3	3	3	
4	4	4			
5					

						1	1	1
			1	1	1	2	2	
			2	2	2	3		
			3	3	4			
		4	4					
3	×	5						

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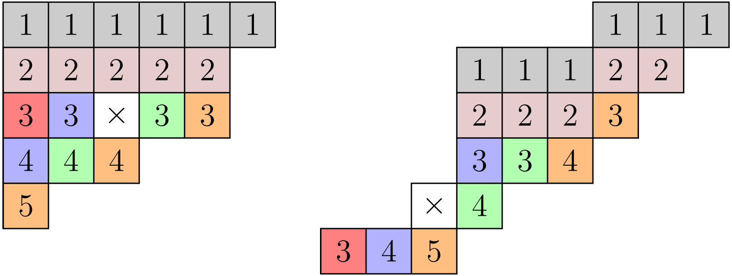

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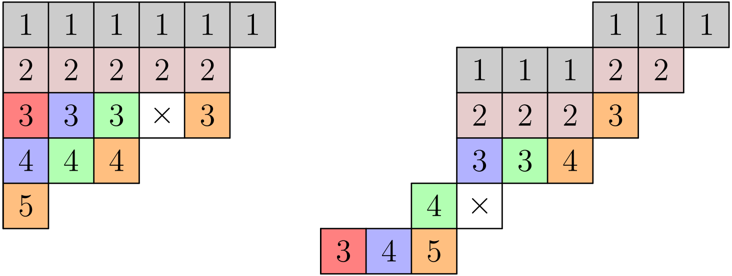

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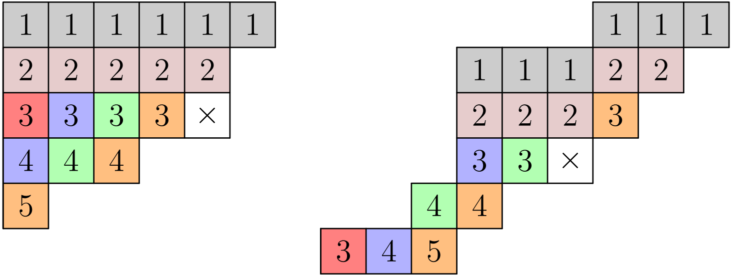

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Connection between ω and K

Theorem (G., Levinson.)

We have two bijections:

$$K(\gamma^c/\alpha;\beta) \leftrightarrow \{\textit{non-adjacent moves in some Phase 1}\}$$

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 & & & & & & 1 & 1 & 1 \\
 & & & \times & 1 & 1 & 2 & 2 & \\
 & & & 1 & 2 & 2 & 3 & & \\
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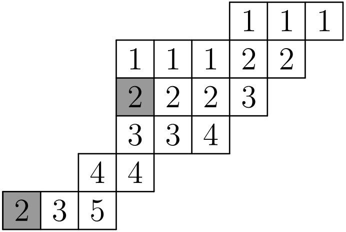

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Proofs of parity and inequality

- We can now give a combinatorial proof of the relations

$$\begin{aligned} K &\geqslant |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{orbits}(\omega)|, \\ K &\equiv \mathrm{sign}(\omega) \pmod{2}. \end{aligned}$$

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- X_i – set of tableaux arising in esh (or sh, equivalently) when the $\boxtimes$ is between the horizontal strips of $(i-1)$'s and i's.
- e_i and s_i – partial steps of esh and sh.

$$
\begin{tikzcd}
X_1 \arrow[r, bend right, "e_1"'] & X_2 \arrow[l, bend right, "s_1"'] \arrow[r, bend right, "e_2"'] & X_3 \arrow[l, bend right, "s_2"'] \arrow[r, bend right, "e_3"'] & \cdots \arrow[l, bend right, "s_3"'] \arrow[r, bend right, "e_t"'] & X_{t+1} \arrow[l, bend right, "s_t"']
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$\omega_i = (s_1 \cdots s_{i-1})(s_i \circ e_i)(s_{i-1}^{-1} \cdots s_1^{-1})$

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- **Key Factorization:**

$$\omega = \mathrm{sh} \circ \mathrm{esh} = (s_1 \cdots s_t) \circ (e_t \cdots e_1) = \omega_t \circ \cdots \circ \omega_2 \circ \omega_1.$$

Therefore $\mathrm{sign}(\omega) \equiv \sum \mathrm{sign}(\omega_i) \equiv \sum \mathrm{sign}(s_i \circ e_i) \pmod 2$.

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The cycles of $s_i \circ e_i$ are "mini-Pieri cases": all steps but one move the $\boxtimes$ down one row in the strip of i's, and the last step returns it to the top of the cycle.

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- An orbit $\mathcal{O}_i$ of $s_i \circ e_i$ generates $|\mathcal{O}_i| - 1$ of the genomic tableaux in Phase 1. So
$$K = \sum_i \left(\sum_{\mathcal{O}_i \in \mathrm{orbits}(\omega_i)} |\mathcal{O}_i| - 1 \right)$$

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- We have

$$\begin{aligned} K &= \sum_i \left(\sum_{\mathcal{O}_i \in \mathrm{orbits}(\omega_i)} |\mathcal{O}_i| - 1 \right) \\ &= \sum_i \mathrm{rlength}(\omega_i) \\ &\geqslant \mathrm{rlength}(\omega) \\ &= |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{orbits}(\omega)| \end{aligned}$$

Here $\mathrm{rlength}(\pi)$ denotes the minimal number of transpositions (reflections) needed to generate π.

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"Elementary!"

Conjectured "orbit-by-orbit" inequality

- Recall that $K \geqslant |\mathrm{LR}(\alpha, \square, \beta, \gamma)| - |\mathrm{orbits}(\omega)|$.

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- Equality holds when $\omega = \mathrm{id}$, implying that in this case S is a disjoint union of complex $\mathbb{P}^1$'s.
- When does equality hold in general?

Constructing high-genus Schubert curves

- **Arithmetic genus:** If ω has exactly one orbit, it turns out that S is connected and integral, and the arithmetic genus is $g(S) = 1 - \chi(\mathcal{O}_S) = K - |\mathrm{LR}(\alpha, \square, \beta, \gamma)| + 1.$
- Arbitrarily high genus: it suffices to find α, β, γ such that ω has exactly one orbit $\mathcal{O}$ and $K \gg |\mathcal{O}| - 1.$

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"The case is one where we have been compelled to reason backward from effects to causes."

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- Arbitrarily high genus: it suffices to find α, β, γ such that ω has exactly one orbit $\mathcal{O}$ and $K \gg |\mathcal{O}| - 1$.
- Let $t \geqslant 2$ be a positive integer. Let $\boxplus = (t+2)^{t+1}$ and set

$$\alpha = \gamma = (t, t-1, t-2, \dots, 2, 1); \qquad \beta = (t+1, 2, 1^{t-2})$$

so γ^c/α is a *staircase-ribbon*. Then ω has one orbit, and the arithmetic genus of S is $g(S) = (t-1)(t-2)$.

$$\begin{array}{cccccccc}
 & & & & & & 1 & 1 \\
 & & & & & 1 & 2 & \\
 & & & & 1 & 2 & & \\
 & & & 1 & 3 & & & \\
 & & 1 & 4 & & & & \\
 & 1 & 5 & & & & & \\
\times & 6 & & & & & &
\end{array}$$

Thank You!

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Proposition (G., Levinson.)

Let $T \in \mathrm{LR}(\alpha, \Box, \beta, \gamma)$. We have $\omega(T) = T$ if and only if the computation of $\mathrm{esh}(T)$ consists only of vertical slides in Phase 1 and horizontal slides in Phase 2.

Thus no fixed point (orbit of length 1) can have genomic tableaux, and so the inequality is an equality for such orbits.

		$\times$	1	1	1
		1	2	2	
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\begin{array}{|c|c|c|c|c|}
\cline{5-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\
\cline{4-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} \\
\cline{3-4}
\multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
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\times & 2 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
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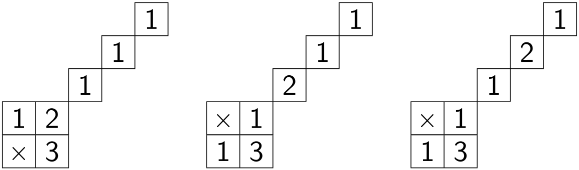

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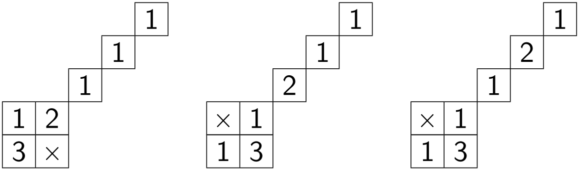

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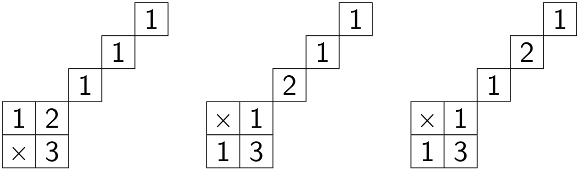

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