Enumerating Anchored Permutations with Bounded Gaps

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Rocky Mountain Algebraic Combinatorics Seminar
Nov 15, 2019

Classical stair climbing problems

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for $n \geqslant 2$.

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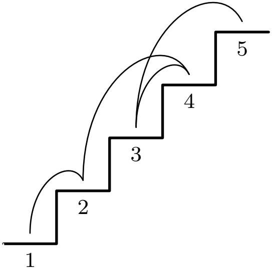
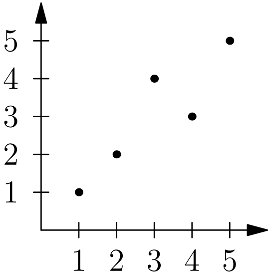
Take up to k stairs at a time? If b_n is number for n stairs:

$$b_n = b_{n-1} + b_{n-2} + \cdots + b_{n-k}$$

Forwards and backwards steps

- Take steps of at most k stairs up or down, stepping on every stair exactly once, starting at stair 1 and ending at stair n?





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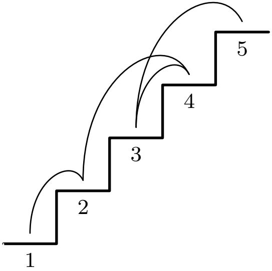
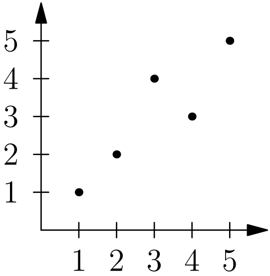
- Permutation $\pi = \pi_1, \ldots, \pi_n$ of stairs $1, \ldots, n$ is **anchored** if $\pi_1 = 1$ and $\pi_n = n$.
- k-**bounded** if $|\pi_i - \pi_{i+1}| \leqslant k$ for all i.





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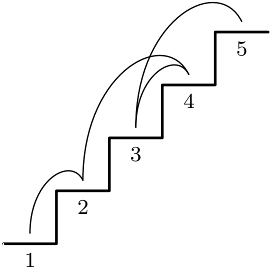
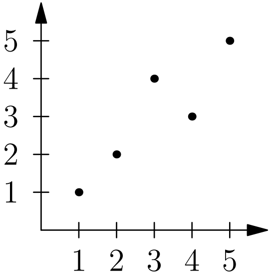
- k-**bounded** if $|\pi_i - \pi_{i+1}| \leqslant k$ for all i.
- Let $F_n^{(k)}$ be number of k-bounded anchored permutations of n. Recursion for $F_n^{(k)}$?





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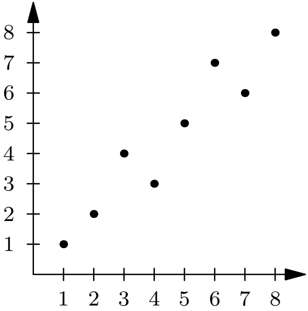

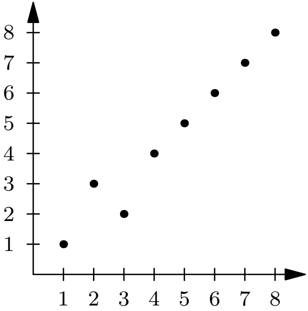

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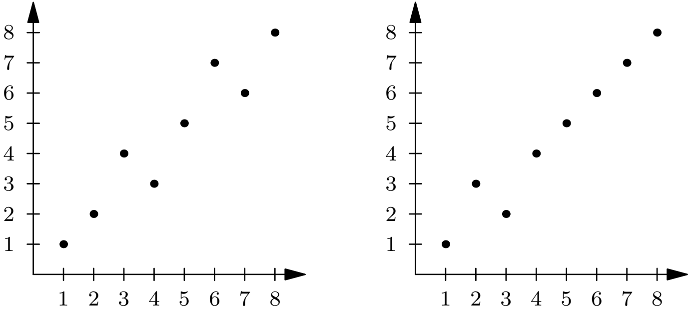

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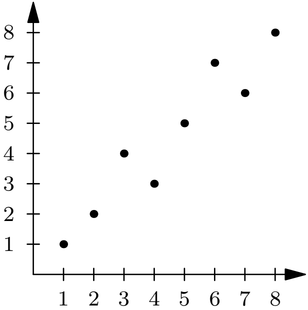
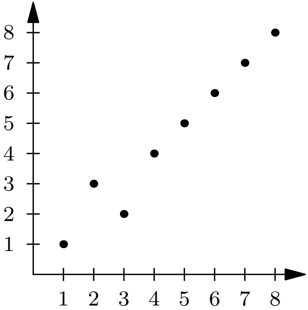
- Recursion:

$$F_n^{(2)} = F_{n-1}^{(2)} + F_{n-3}^{(2)}$$





The case $k = 3$

Define $F_n = F_n^{(3)}$.

- Using generating functions:

$$F_n = 2F_{n-1} - F_{n-2} + 2F_{n-3} + F_{n-4} + F_{n-5} - F_{n-7} - F_{n-8}$$

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- **Solved conjecture listed on OEIS!** Screenshot from 2018:

A249665 The number of permutations p of {1,...,n} such that p(1)=1, p(n)=n, and |p(i)-p(i+1)| is in {1,2,3} for all i from 1 to n-1.

```
1, 1, 1, 2, 6, 14, 28, 56, 118, 254, 541, 1140, 2401, 5074, 10738, 22711, 48001, 101447,
214446, 453355, 958395, 2025963, 4282685, 9053286, 19138115, 40456779, 85522862, 180789396,
382176531, 807895636, 1707837203, 3610252689, 7631830480
```
(list; graph; refs; listen; history; text; internal format)

OFFSET 1,4

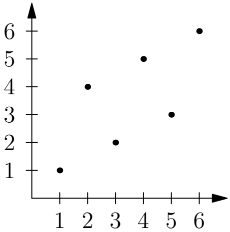
LINKS Andrew Woods, Table of n, a(n) for n = 1..250

FORMULA

```
Let a(1)=1, g(1)=h(1)=0. For all n<1, let a(n)=g(n)=h(n)=0. Then:
a(n) = a(n-1) + g(n-1) + h(n-1),
g(n) = a(n-2) + a(n-3) + a(n-4) - a(n-6) + g(n-2) + g(n-4) + h(n-2),
h(n) = 2*a(n-3) + 2*a(n-4) + a(n-5) - a(n-7) + g(n-3) + g(n-5) + h(n-3).
Alternatively, let a(1)=1, a(n)=0 for n<1. Let b(1)=1, b(2)=0, b(3)=1,
  b(4)=3, b(5)=4, b(6)=5, b(7)=7, b(8)=10, and b(n)=b(n-1)+b(n-3) for n>8.
  Then:
a(n) = a(n-1)*b(1) + a(n-2)*b(2) + a(n-3)*b(3) + ... + a(1)*b(n-1).
Conjectures from Colin Barker, Mar 07 2015: (Start)
a(n) = 2*a(n-1)-a(n-2)+2*a(n-3)+a(n-4)+a(n-5)-a(n-7)-a(n-8).
G.f.: -x*(x^3+x-1) / (x^8+x^7-x^5-x^4-2*x^3+x^2-2*x+1).
(End)
```

The case $k = 3$: Proof

- **Joker**: 31425, or any vertical translation thereof appearing consecutively



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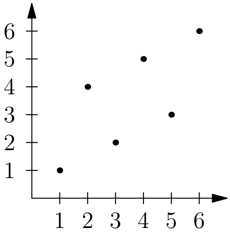

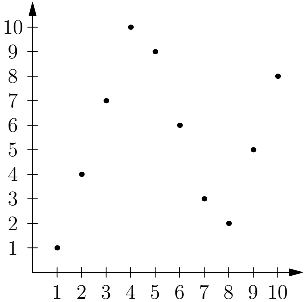

- **Lemma:** (G.,M.,M.) At (i, i) after permutation of $1, 2, \ldots, i$: up-step of $+3$ must be followed either by a **Joker** or by a **Cascading** 3-**pattern**.

The case $k = 3$: Proof

- $F_n = \#$ 3-bounded anchored permutations of length n
- $G_n = \#$ 3-bounded permutations π with $\pi_1 = 1$ or $\pi_1 = 2$ and $\pi_n = n$
- $H_n = \#$ 3-bounded permutations π with $\pi_1 = 3$ and $\pi_n = n$ that do not begin with the Joker

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- System of recursions:
  1. $F_n = G_{n-1} + H_{n-1} + F_{n-5}$
  2. $G_n = F_n + G_{n-2} + F_{n-3} + G_{n-4} + H_{n-2}$
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  3. $H_n = F_{n-3} + G_{n-3} + F_{n-4} + G_{n-5} + H_{n-3}$
- Set $F(x), G(x), H(x)$ to be generating functions of F_n, G_n, H_n. Solve system of three equations:

$$F(x) = \frac{x - x^2 - x^4}{1 - 2x + x^2 - 2x^3 - x^4 - x^5 + x^7 + x^8}$$

Recursion follows. QED

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- **Transfer-matrix method:** to show gen. function is rational
- **Finite directed graph:** (V, E) where $E \subseteq V \times V$
- **Adjacency matrix:** For $i, j \in V$, define $A_{ij} = \begin{cases} 1 & (i,j) \in E \\ 0 & (i,j) \notin E \end{cases}$

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Theorem (Transfer-matrix.)

Let $p_{ij}(n) = \#$ directed paths from i to j of length n. Then

$$\sum_{n=0}^{\infty} p_{ij}(n) x^n = \frac{(-1)^{i+j} \det(I - xA; j, i)}{\det(I - xA)} \in \mathbb{C}(x)$$

where $\det(B; j, i)$ is the minor with row j, column i deleted.

Transfer-matrix method for non-anchored case

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Example

$\pi = 51432$ has consecutive 3-patterns:

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- **Pattern graph $\mathcal{P}_k$:** nodes are k-patterns, edge $\tau \to \sigma$ iff pattern of $\tau_2, \ldots, \tau_k$ matches pattern of $\sigma_1, \ldots, \sigma_{k-1}$

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Path of 51432 in $\mathcal{P}_3$ is $312 \to 132 \to 321$

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- **k-determined permutation:** determined by its path of consecutive patterns in $\mathcal{P}_k$

Example

$\pi = 51432$ has consecutive 3-patterns: 312, 132, 321
Path of 51432 in $\mathcal{P}_3$ is $312 \to 132 \to 321$
Path of 52431 in $\mathcal{P}_3$ is $312 \to 132 \to 321$, so not 3-determined

Transfer-matrix method for non-anchored case

Theorem (Avgustinovich, Kitaev, 2008)

For any permutation π:

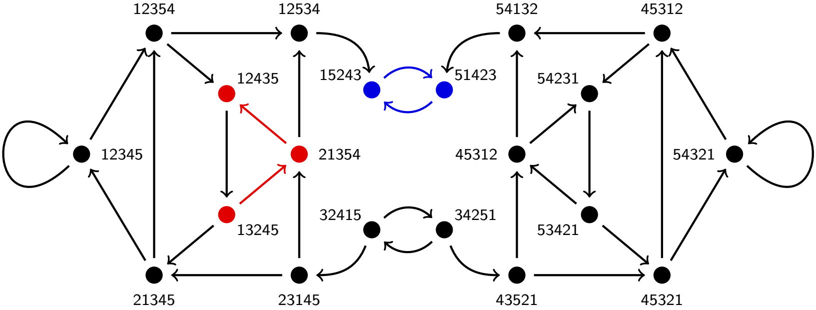
$$\begin{aligned}\pi \text{ is } (k+1)\text{-determined} &\Leftrightarrow \pi^{-1} \text{ is } k\text{-bounded}\\ &\Leftrightarrow \pi \text{ avoids all } k\text{-prohibited}\\ &\qquad \text{patterns of length at most } 2k+1\end{aligned}$$

A k-**prohibited pattern** is of the form $xX(x+1)$ or $(x+1)Xx$ where $|X| \geqslant k$.

Definition

$\mathcal{P}_{2k+1,k}$ is the subgraph of $\mathcal{P}_{2k+1}$ on nodes that do not contain a k-prohibited pattern.

Example: $\mathcal{P}_{5,2}$

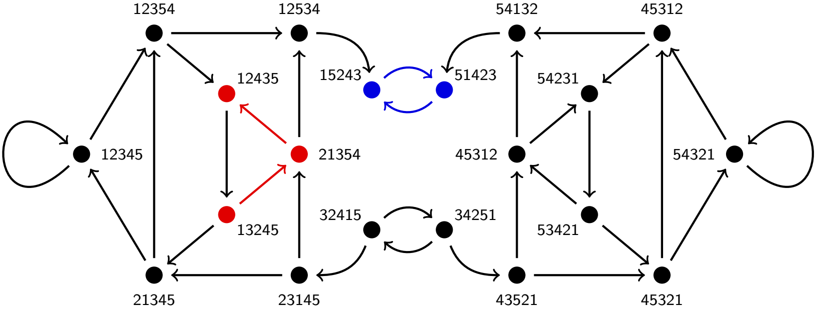


(Paths of length $n-4$) $\longleftrightarrow$ (2-bounded permutations of length n)

(13245 $\rightarrow$ 21354 $\rightarrow$ 12435) $\longleftrightarrow$ 1324657 (inverse of 1324657)

(15243 $\rightarrow$ 51423 $\rightarrow$ 15243) $\longleftrightarrow$ 1726354 (inverse of 1357642)

Example: $\mathcal{P}_{5,2}$



Lemma (G.,M.,M.)

An inverse k-bounded permutation π is anchored **if and only if** *its first consecutive pattern of length $2k+1$ starts with 1 and its last ends with $2k+1$.*

Anchored permutations as paths in $\mathcal{P}_{2k+1,k}$

Theorem (G.,M.,M.)

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By the Transfer-Matrix theorem:

$$F^{(k)}(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{p(x)}{\det(I - xA)}$$

where A is adjacency matrix of $\mathcal{P}_{2k+1,k}$, $p(x)$ is some polynomial.

Consequences

$$F^{(k)}(x) = \sum_{n=0}^{\infty} F_n^{(k)} x^n = \frac{p(x)}{\det(I - xA)}$$

- A finite-depth linear recurrence for $F_n^{(k)}$ exists for all k!

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- Let $\{\alpha_i\}$ be eigenvalues of A with multiplicities $\{d_i\}$. Characteristic poly $\det(xI - A)$ factors as $\prod_i (x - \alpha_i)^{d_i}$, so

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- Use partial fractions and expand the generating function:

$$F_n^{(k)} = \sum_i p_i(n) \alpha_i^n$$

for some polynomials $p_i(n)$ of degree at most $d_i - 1$.

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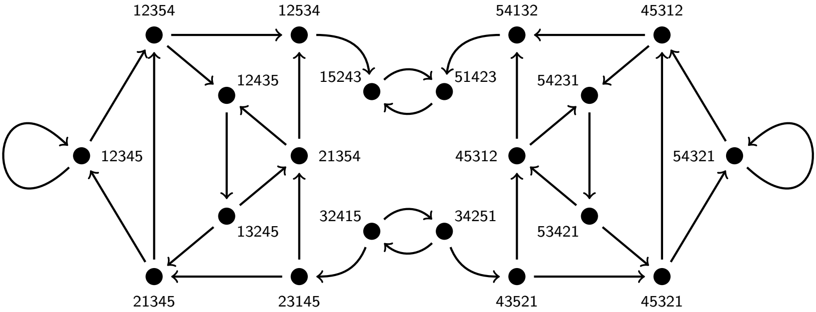
Theorem (Frobenius)

If digraph is **strongly connected**, *eigenvalues of adjacency matrix are bounded above by the max outdegree of any vertex.*



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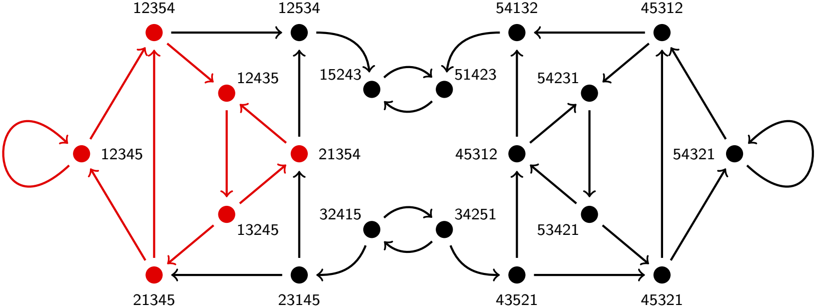

Lemma (G.,M.,M.)

All paths in $\mathcal{P}_{2k+1,k}$ that correspond to anchored permutations lie in the strongly connected component $\mathcal{P}'_{2k+1,k}$ of the identity.

Asymptotics

Theorem (Perron-Frobenius)

For an adjacency matrix A of a strongly connected digraph having at least one loop, there is a **unique** *eigenvalue r of maximal absolute value, r has multiplicity 1, and r is a positive real number.*

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Corollary

For some constant c and polynomials p_i,

$$F_n^{(k)} = c \cdot r^n + \sum_{\alpha_i \neq r} p_i(n)\alpha_i^n,$$

where $|\alpha_i| < r$ for all other eigenvalues α_i. Hence $F^{(k)}(n)$ is $O(r^n)$ (asymptotically bounded above by cr^n for some c.)

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Theorem (G.,M.,M.)

We have that $F_n^{(k)}$ is $O(k^n)$.

Proof: Maximum outdegree in $\mathcal{P}'_{2k+1,k}$ is k.

Asymptotics: Sanity check

Theorem (G.,M.,M.)

We have that $F_n^{(k)}$ is $O(k^n)$.

- $\mathbf{k} = \mathbf{2}$: largest root of $x^3 - x^2 - 1$ is approximately

$$r \approx 1.466 < 2$$

so $F_n^{(2)}$ is $O(2^n)$.

Asymptotics: Sanity check

Theorem (G.,M.,M.)

We have that $F_n^{(k)}$ is $O(k^n)$.

- **k = 2**: largest root of $x^3 - x^2 - 1$ is approximately

$$r \approx 1.466 < 2$$

  so $F_n^{(2)}$ is $O(2^n)$.

- **k = 3**: largest root of $x^8 - 2x^7 + x^6 - 2x^5 - x^4 - x^3 + x + 1$ is

$$r \approx 2.114 < 3$$

  so $F_n^{(3)}$ is $O(3^n)$.

Summary

1. How many ways to climb n stairs with steps of size $\pm 1, \pm 2, \pm 3$? Satisfies recursion

$$F_n = 2F_{n-1} - F_{n-2} + 2F_{n-3} + F_{n-4} + F_{n-5} - F_{n-7} - F_{n-8}$$

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3. Stair climbs correspond to paths in a certain strongly connected component of a pattern overlap graph.

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1. How many ways to climb n stairs with steps of size $\pm 1, \pm 2, \pm 3$? Satisfies recursion

$$F_n = 2F_{n-1} - F_{n-2} + 2F_{n-3} + F_{n-4} + F_{n-5} - F_{n-7} - F_{n-8}$$

2. By showing that the generating functions are rational: such a recurrence exists for steps $\pm 1, \pm 2, \ldots, \pm k$ for all k.
3. Stair climbs correspond to paths in a certain strongly connected component of a pattern overlap graph.
4. Number of stair climbs for k is asymptotically bounded above by $c \cdot k^n$ for some constant c.

Thank You!