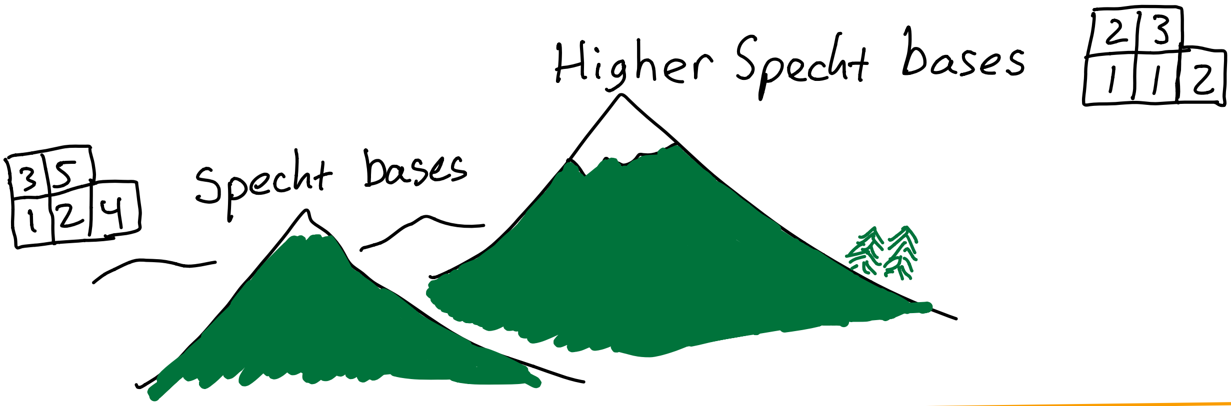

Higher Specht bases for generalizations of the coinvariant ring

Rocky Mtn. Algebraic Combinatorics

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On joint work with

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BACKGROUND: S_n-modules and the coinvariant ring

Def: ① S_n = group of permutations of $\{1, \dots, n\}$
"Symmetric group".

② S_n-module: a $\mathbb{C}$-vector space V with an action of S_n by linear transformations.

Ex: $P_n := \mathbb{C}[x_1, \dots, x_n]$

Action $S_n \curvearrowright P_n$ by:

$$\pi f(x_1, \dots, x_n) = f(x_{\pi(1)}, \dots, x_{\pi(n)})$$

$$(12)\cdot(x_1^2 x_2 + 2x_1 + x_3^4) = x_2^2 x_1 + 2x_2 + x_3^4$$

Def: submodule of an S_n-module V is an S_n-invariant subspace $W \subseteq V$.

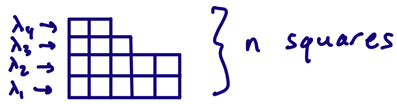
Irreducible: if no proper nonzero submodules

Decomposition: $V = W \oplus U$ W, U submodules

S_n-MODULES vs POSITIVE INTEGERS

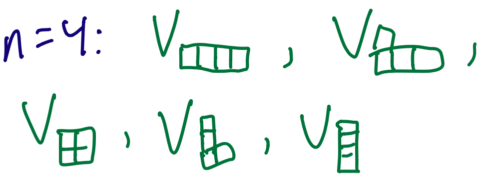
Irreducible — Prime

V_λ for each partition $\lambda = (\lambda_1, \lambda_2, \dots)$ of n



$p_1, p_2, p_3, p_4, \dots$

$n=4$:



$2, 3, 5, 7, \dots$

Decomposition into irreducibles — Prime factorization

$$V \cong \bigoplus_{\lambda \vdash n} c_\lambda V_\lambda \qquad\qquad m = \prod p_i^{\alpha_i}$$

Def: Irreducibles V_λ are called <u>Specht modules</u>.

Construction: $V_\lambda \subseteq P_n$ generated by "Specht polynomials":

Def: <u>Standard Young tableau</u> T of shape λ is filling with $1, 2, \dots, n$ s.t. rows, cols are increasing.

$SYT(n) = \{ \text{SYT's } T \text{ of size } n \}$ $\qquad SYT(\lambda) = \{ T \in SYT(n) \text{ shape } \lambda \}$

Ex:

$$T = \begin{array}{|c|c|c|c|c|} \hline 7 \\ \cline{1-1} 2 & 5 & 8 & 9 \\ \cline{1-4} 1 & 3 & 4 & 6 & 10 \\ \hline \end{array}$$

$$F_T = (x_7 - x_2)(x_7 - x_1)(x_2 - x_1) \cdot (x_5 - x_3)(x_8 - x_4)(x_9 - x_6)$$

Def:

$$F_T = \prod_{\substack{C \text{ column} \\ \text{of } T}} \prod_{i<j \in C} (x_j - x_i)$$

("Product of Vandermonde determinants of columns")

$$\boxed{\text{FACT: } \{F_T : T \in SYT(\lambda)\} \text{ spans } V_\lambda \subseteq \mathbb{C}[x_1, \dots, x_n].}$$

Ex: $V_{\ydiagram{4}}$ spanned by $F_{\ytableaushort{1234}} = 1$ (trivial rep)

$V_{\ydiagram{1,1,1,1}}$ spanned by $F_{\ytableaushort{4,3,2,1}}$ = vandermonde (sign rep)

$V_{\ydiagram{2,2}}$ spanned by Specht polynomials

$$F_{\ytableaushort{34,12}} = (x_3-x_1)(x_4-x_2) \quad \text{AND} \quad F_{\ytableaushort{24,13}} = (x_2-x_1)(x_4-x_3)$$

Q: How to decompose P_n into irreducibles?

- $P_n = \mathbb{C}[x_1,\dots,x_n]$ infinite-dimensional $\Rightarrow$ Some V_λ occur infinitely many times
- One way to reduce to finite dimensions:

$$P_n = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1,\dots,x_n]_{(d)}$$

degree-d homogeneous part

- Better way: quotient by all trivial reps in deg $d \geq 1$.
  (symmetric functions)

Def: Coinvariant ring $R_n = \mathbb{C}[x_1,\dots,x_n]/Sym_+$

$$= \mathbb{C}[x_1,\dots,x_n]/(e_1,\dots,e_n)$$

where $e_d(x_1,\dots,x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}$

Ex: $e_2(x_1,x_2,x_3) = x_1x_2 + x_1x_3 + x_2x_3$

FACTS:
1. R_n has dimension $n!$ as a $\mathbb{C}$-vector space
2. R_n is regular representation of S_n
3. $P_n = R_n \otimes \Lambda_n$ where $\Lambda_n = \mathbb{C}[x_1,\dots,x_n]^{S_n}$
4. $R_n \cong H^*(Fl_n)$ where Fl_n = complete flag variety in $\mathbb{C}^n$

Monomial basis for R_n: $\{ x_1^{a_1} \cdots x_n^{a_n} : a_i \leq n-i \}$

↳ Has $n!$ elements, but doesn't respect decomposition into irreducibles.

Visual: $V = W \oplus U$

Ex: $R_2 = \mathbb{C}[x_1, x_2]/(x_1 + x_2,\ x_1 x_2) \cong \mathbb{C}[x_1]/(-x_1^2) \cong \mathbb{C}[x_1]/(x_1^2)$

Basis: $1,\ x_1$

Better basis: $1,\ x_2 - x_1$

$R_2 = V_{\square\square} \oplus V_{\substack{\square \\ \square}}$

$\boxed{\text{Q: Is there a basis for } R_n \text{ that respects its decomposition into irreducibles?}}$

Ariki, Terasoma, Yamada (2005): Yes! "Higher Specht basis"

QUESTIONS BREAK

Ariki, Terasoma, and Yamada's Construction

Def: Let $S \in SYT(n)$. A <u>descent</u> of S is an entry i s.t. $i+1$ is in a higher row than i.
Write $\mathrm{des}(S) = \#$ descents.

Ex: $S = \begin{array}{|c|c|c|} \hline 4 \\ \cline{1-1} 3 & 6 & 7 \\ \hline 1 & 2 & 5 \\ \hline \end{array}$ $\quad \mathrm{des}(S) = 3 \quad$ Labels: $\begin{array}{|c|c|c|} \hline 2 \\ \cline{1-1} 1 & 3 & 3 \\ \hline 0 & 0 & 2 \\ \hline \end{array}$

Def: <u>Cocharge labeling</u> of S: label entry i with the number of descents less than i.

Def: Let $S, T \in SYT(\lambda)$ for some λ. Then

$$x_T^S = \prod_{\substack{s \text{ square} \\ \text{in } \lambda}} x_{T(s)}^{\text{cocharge label of } S(s)}$$

Ex: $T = \begin{array}{|c|c|c|} \hline 6 \\ \cline{1-1} 2 & 5 & 7 \\ \hline 1 & 3 & 4 \\ \hline \end{array}$ $\quad x_T^S = x_1^0 x_2^1 x_3^0 x_4^2 x_5^3 x_6^2 x_7^3$

Pairs (S,T) same shape $\overset{RSK}{\longleftrightarrow}$ Perms of $1, \dots, n$

Def: The <u>higher Specht polynomial</u> F_T^S is

$$F_T^S = \sum_{\tau \in Col(T)} \sum_{\sigma \in Row(T)} \mathrm{sgn}(\tau)\, \tau \sigma\, x_T^S$$

where $Row(T)$ and $Col(T)$ are the groups of row and column permutations of T respectively.

Note: For $S = \begin{array}{|c|c|c|} \hline 7 \\ \cline{1-1} 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array}$ $\left(\text{labels} = \begin{array}{|c|c|c|} \hline 2 \\ \cline{1-1} 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ \hline \end{array} \right)$

$$F_T^S = |Row(T)| \cdot \underbrace{F_T}_{\text{ordinary Specht polynomial}}$$

For other S: "higher degree" versions.

Thm: (Ariki, Terasoma, Yamada) Fix $S \in SYT(\lambda)$. Then

$$\{F_T^S : T \in SYT(\lambda)\}$$

spans a copy of V_λ.

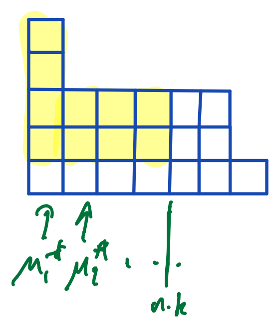
Moreover, the union of these sets over all S is a basis for R_n. ("higher Specht basis").

GENERALIZATIONS

① The Garsia-Procesi Modules R_μ

Def: $R_\mu = \mathbb{C}[x_1, \dots, x_n] \Big/ \Big(e_r(x_{i_1}, \dots, x_{i_k}) : r > \overbrace{\mu_1^* + \dots + \mu_{n-k}^*}^{\text{first } n-k \text{ cols}} - (n-k) \Big)$

- $R_{1^n} = R_n$
- $R_{(n)} = \mathbb{C}$

$\mu =$ 

- In general R_μ is a quotient of R_n, admits $S_n \curvearrowright R_\mu$
- $R_\mu \cong H^*(X_\mu)$ where $X_\mu = \{$complete flags in $\mathbb{C}^n$ fixed by unipotent matrix of Jordan type $\mu\}$

$X_\mu =$ "Springer fiber"

Def: Semistandard Young tableau (SSYT) of shape λ is filling of λ w/ pos. ints. s.t.
- rows weakly increase
- cols strictly increase.

Def: Content of an SSYT S is (#1's, #2's, ...)

Ex: $S =$

3	4	5		
2	2	3	4	
1	1	1	1	2

Content $= (4, 3, 2, 2, 1)$

labels:

2	2	3		
1	1	1	2	
0	0	0	0	0

Def: Cocharge labels: (a) search backwards in reading order (cyclically) for a 1, then 2, then 3, ...
(b) Label that standard subword with cocharge labels
(c) Iterate this process on remaining labels.

Def: For $T \in SYT(\lambda)$, $S \in SSYT(\lambda)$,

$$x_T^S = \prod_{\substack{s \text{ square} \\ \text{in } \lambda}} x_{T(s)}^{\text{cocharge label of } S(s)}$$

semistandard

$$F_T^S = \sum_{\tau \in Col(T)} \sum_{\sigma \in Row(T)} sgn(\tau)\tau\sigma\, x_T^S .$$

Conj. (G., Rhoades) The set

$$B_\mu = \left\{ F_T^S : \begin{array}{l} S \in SSYT(\lambda),\ T \in SYT(\lambda) \text{ for} \\ \text{some } \lambda,\ S \text{ has content } \mu \end{array} \right\}$$

is a higher Specht basis for R_μ.

Thm. (G., Rhoades)

(1) If μ has two rows, then B_μ is a higher Specht basis for R_μ.

(2) For any μ, if B_μ is a basis of R_μ, then it is a higher Specht basis. In particular, for a fixed SSYT S_0 of content μ and shape λ, if the set $\{F_T^{S_0} : T \in SYT(\lambda)\}$ is independent in R_μ then it spans a copy of V_λ.

Proof of (1): uses a known basis to compare to, induction on $|\mu|$.

Proof of (2): standard techniques from S_n-rep. theory

② The Haglund-Rhoades-Shimozono modules $R_{n,k}$

Def: $R_{n,k} = \mathbb{C}[x_1,\dots,x_n] / (x_1^k, x_2^k, \dots, x_n^k, e_{n-k+1}, \dots, e_n)$

- $R_{n,n} = R_n$ • $R_{n,1} = \mathbb{C}$
- Appears in $t=0$ case of "Delta Conjecture"
- $R_{n,k} \cong H^*(X_{n,k})$ where $X_{n,k}$ is "line configuration space"

$$X_{n,k} = \left\{ (\ell_1, \dots, \ell_n) : \begin{array}{l} \ell_i \text{ line in } \mathbb{C}^k \\ \ell_1 + \dots + \ell_n = \mathbb{C}^k \end{array} \right\}$$

THM (G., Rhoades) The set

$$B_{n,k} = \{ F_T^S \cdot e_1^{i_1} \cdots e_{n-k}^{i_{n-k}} : F_T^S \in B_n,\ i_1 + \dots + i_{n-k} < k - des(S) \}$$

is a higher Specht basis of $R_{n,k}$. 😎

Proof used further generalization

$$R_{n,k,s} = \mathbb{C}[x_1,\dots,x_n] / (e_n, \dots, e_{n-s+1}, x_1^k, \dots, x_n^k)$$

and exact sequence

$$0 \to R_{n,k-1,s} \xrightarrow{\cdot e_{n-s}} R_{n,k,s} \xrightarrow{e_{n-s} \to 0} R_{n,k,s+1} \to 0$$

Started at $s=0$, induct on s for any fixed k.

At $k=s=n$, get R_n, basis is $\{F_T^S\}$

Cor: (G., Rhoades) New inductive proof of ATY's result on higher specht basis for R_n.

③ The Griffin modules $R_{n,k,\mu}$ (Apr 2020)

- Common generalization of $R_{n,k}$ and R_μ:

$$R_{n,k,(1^k)} = R_{n,k}$$

$$R_{n,k,\mu} = R_\mu \quad \text{if } |\mu| = n, \ k \geq \ell(\mu)$$

- Coordinate rings of certain "rank varieties"
- $R_{n,k,\mu} \cong H^*(???)$
- S_n-module structure not yet fully understood

THM: (G., Rhoades) A higher Specht basis for

$$R_{n,k,\underbrace{(n-1)}_{n-1}}$$

is given by

$$\left\{ F_T^S \cdot e_1^i \ : \ F_T^S \in B_{(n-1,1)}, \ i < k - \mathrm{des}(S) \right\}$$

😎

PROOF uses SES

$$0 \to R_{n,k,\mu} \xrightarrow{\cdot e_{n-|\mu|}} R_{n,k,\underline{\mu}} \xrightarrow{e_{n-|\mu|} \to 0} R_{n,k+1,\underline{\mu+(1)}} \to 0$$

and our theorems for R_μ.