Groups, Lie groups, Lie algebras and their representations

Last time: Correspondence:

$$\left(\begin{array}{c}\text{connected Lie}\\ \text{groups}\end{array}\right) \longleftrightarrow \left(\text{Lie algebras}\right)$$

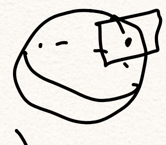
$$\text{Reps} \longleftrightarrow \underbrace{\text{Reps}}_{\text{Lots of nice combinatorics}}$$

Lie algebra of a Lie group G is $T_e G$

(Abstract Lie algebra): Vector space V w/ a Lie bracket $[\,,\,] : V \times V \to V$ s.t. • bilinear,
1) $[X,Y] = -[Y,X]$
2) Jacobi: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$

Ex: $sl_n(\mathbb{C}) = \{X \in Mat_n(\mathbb{C}) : \operatorname{tr} X = 0\}$ $\qquad \dim_{\mathbb{C}}(sl_3(\mathbb{C})) = 3$

Bracket: $[X,Y] = XY - Y \cdot X$



$\boxed{\text{Today: Representations of } sl_2(\mathbb{C})}$

$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl_2(\mathbb{C})$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}$

$\dim_{\mathbb{C}}(sl_2(\mathbb{C})) = 3$

Basis vectors:

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH + bE + cF$$

Recall: Rep of Lie algebra $sl_2(\mathbb{C})$ is a linear map $sl_2(\mathbb{C}) \longrightarrow gl_n(\mathbb{C})$ for some n that sends $[\,,\,] \longrightarrow [\,,\,]$

Brackets of E, F, H:

- $[H,H] = 0 = [E,E] = [F,F]$
- $[H,E] = HE - EH = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = 2E$ *
- $[H,F] = -2F$ *
- $[E,F] = H$

$[E,H] = -[H,E] = -2E$

Bilinear:

$[E+F, H] = [E,H] + [F,H] = -2E + 2F$

Reps of $sl_2(\mathbb{C})$:

$$\rho : sl_2(\mathbb{C}) \to gl(V) \qquad V \cong \mathbb{C}^n$$
$$H \mapsto h = \rho(H)$$

$\boxed{\text{diagonal matrix w.r.t some basis of } V}$

h "acts diagonalizably" on V (by multipication)

Let $V_\alpha = \{ v \in V : hv = \alpha v \}$ (weight space for α)

Thm: $\boxed{V = \bigoplus_{\alpha \in \mathbb{C}} V_\alpha}$ $\qquad V_\alpha$: lines spanned by each eigenvector.

Pf: The fact that h acts diagonalizably is HARD (Appendix C in Fulton-Harris' Rep theory)

Uses Lie bracket preservation!

$[\mathfrak{g},\mathfrak{g}], [\mathfrak{g},[\mathfrak{g},\mathfrak{g}]], \ldots$

$\square$

Cor: If V is a finite-dimensional rep of sl_2 (dim n)

then $V = \bigoplus_{i=1}^{n} V_{\alpha_i}$ for some

$\alpha_1, \ldots, \alpha_n$.

(Each V_{α_i} one-dimensional eigenspace of H)

Thm (restated): If $\boxed{Hv_\alpha = \alpha v_\alpha}$ then $H(Ev_\alpha) = (2+\alpha)Ev_\alpha$

Pf: $HEv_\alpha = (\underbrace{HE - EH} + EH)v_\alpha$

$= ([H,E] + EH)v_\alpha$

$$= [H,E]v_\alpha + E\,Hv_\alpha$$
$$= 2Ev_\alpha + E(\alpha v_\alpha)$$
$$= (2+\alpha)Ev_\alpha.$$

QED.

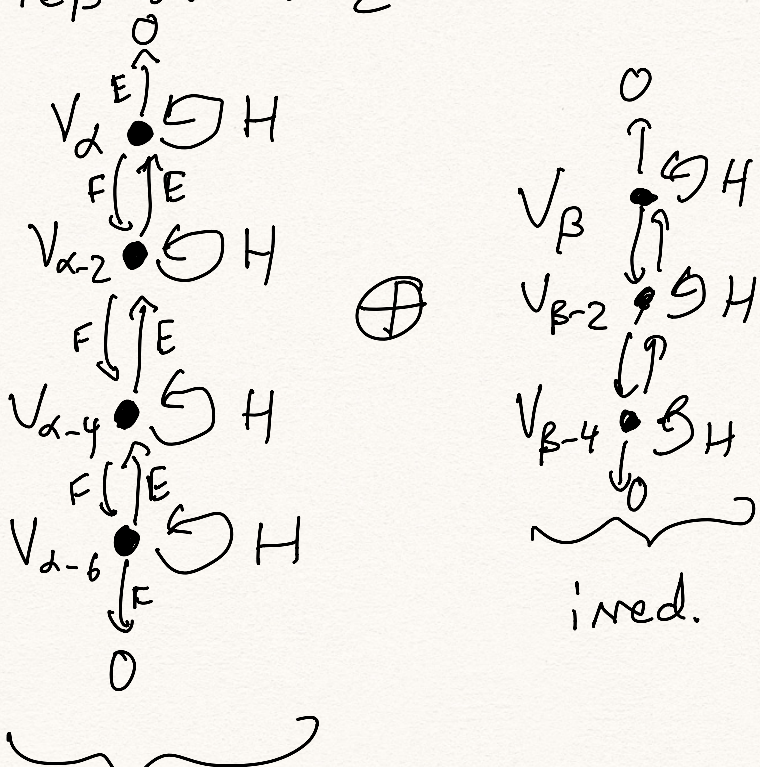
Thm: If $Hv_\alpha = \alpha v_\alpha$ then $H(Fv_\alpha) = (\alpha-2)Fv_\alpha$.

Pf: Exercise.

$[H,F] = -2F$

Cor: If $v \in V_\alpha$, $Fv \in V_{\alpha-2}$ and $Ev \in V_{\alpha+2}$

Picture: A rep of sl_2 looks like $\bigoplus V_\alpha$:



irreducible

irred.

Lie alg reps: linear maps

$$\rho : \mathfrak{g} \longrightarrow \underbrace{\mathfrak{gl}(V)}_{\text{all linear transformations } V \to V}$$

that preserve $[\,,\,]$.

$$\mathfrak{g} \longrightarrow \underline{\mathfrak{gl}_n(\mathbb{C})}$$

$$\begin{array}{l} \,\,\cup\!\!\!\!\shortmid \\ H \longmapsto h \end{array}$$

is a matrix acting on $\mathbb{C}^n$