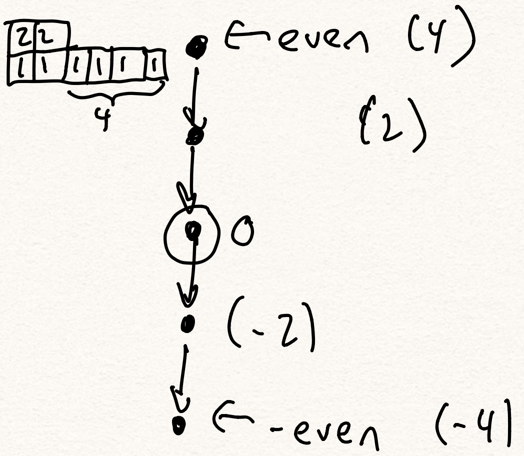
Homework A problem 7(a):

Show # ballot sequences of 1's and 2's of length $2n$ is $\binom{2n}{n}$ and of length $2n+1$ is $\binom{2n+1}{n+1}$.

Proof using sl_2 chains: (2n case first)

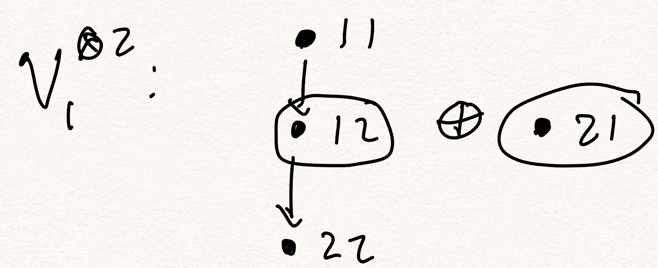
Ballot sequences correspond to highest wt elts of each sl_2 chain in $V_1^{\otimes 2n}$.



Every chain in $\boxed{V_1^{\otimes 2n}}$ has even highest weight
$\Rightarrow$ every chain has a unique weight-0 elt.
(corresponds to a word having n 1's and n 2's)

Moreover, every word having n 1's and n 2's is in a unique chain in $\underbrace{V_1^{\otimes 2n}}_{\{\text{all words in 1's, 2's}\}}$

$V_1^{\otimes 2}$:



words w/ n 1's and n 2's $= \binom{2n}{n}$

Odd case: all weights are odd:

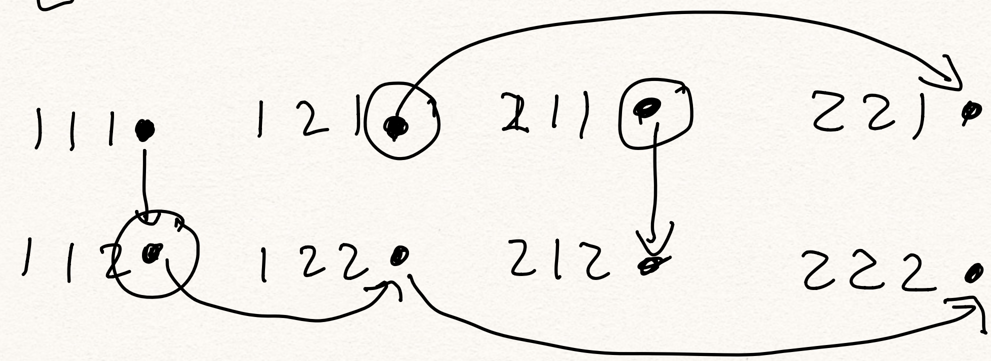
$\Rightarrow$ Every chain has an elt of weight 1

$(n+1 \text{ 1's}, \; n \text{ 2's})$

$\# = \binom{2n+1}{n+1}$

Ex: V_1:

$V_1^{\otimes 3}$:

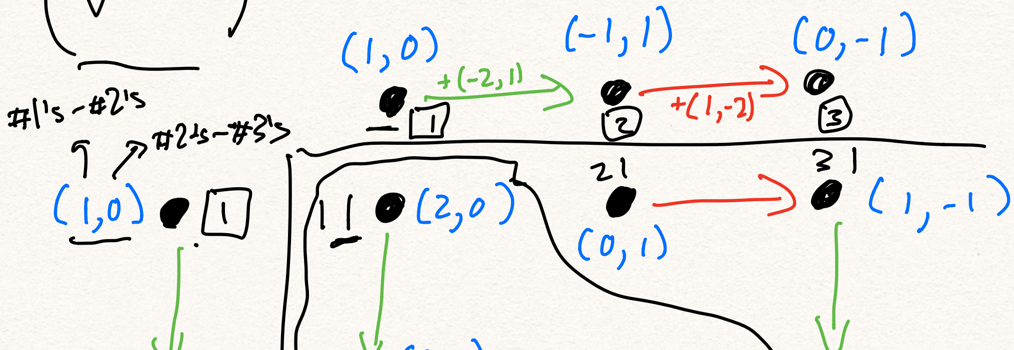


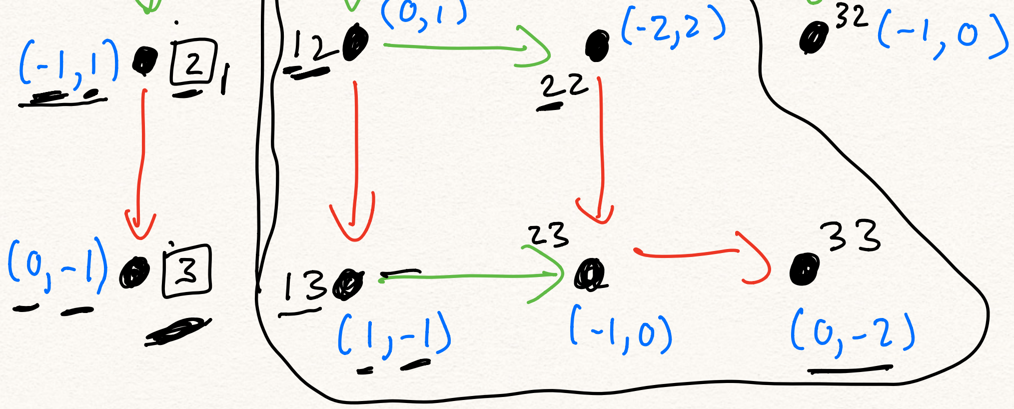
Def: $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$ (or simply $a_1 a_2 \cdots a_n$) denotes the weight space corresponding to $\boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_n}$ in $V_{(1,0)}^{\otimes n}$

Thm: (Goal): Every irred sl_3-rep $V^{(a,b)}$ is a summand of some $\left(V^{(1,0)}\right)^{\otimes n}$ for some n.

(i.e. we can just use word/tableau combinatorics)

$\left(V^{(1,0)}\right)^{\otimes 2}$





Recall: sl_2 weight is eigenvalue of H.

sl_3 weight is eigenvalues (H_1, H_2)

$$H(v \otimes w) = Hv \otimes w + v \otimes Hw$$
$$= (\alpha + \beta)\, v \otimes w$$

(set $H = H_1$ or H_2, each eigenvalue adds)

Lemma: $wt(a_1 a_2 \cdots a_n)$ is $(\#1\text{'s} - \#2\text{'s},\ \#2\text{'s} - \#3\text{'s})$

Pf: By induction, and additivity of weights. $\square$

Lemma: $F_1 := E_{21}$ applied to $a_1 \cdots a_n \in \{1,2,3\}^n$ is the word formed by bracketing 2's and 1's
( )
and changing rightmost unpaired 1 to 2

$F_2 := E_{32}$ applied to $a_1 \cdots a_n$ is formed by bracketing 3's w/ 2's and changing
( )
rightmost unpaired 2 to 3.

Pf: By induction. Base case:

$V^{(1,0)}$:

$$\begin{tikzcd} \bullet\, 2 \arrow[d, "F_2"'] \\ \bullet\, 3 \end{tikzcd}$$

Induction step: Assume true for $V_{(1,0)}^{\otimes(n-1)}$.

Tensor w/ $\underline{V_{(1,0)}}$ to add new letter $(1,2,3)$

Then F_1 structure given by tensoring $V_{(1,0)}^{\otimes(n-1)}$ as sl_2-module with

$$\begin{tikzcd} \boxed{1} \arrow[d] \\ \boxed{2} \end{tikzcd} \oplus \boxed{3}$$

Adding a 3 @ end of word is tensoring w/ trivial $\Rightarrow$ no change in how F_1 applies.

Adding 1's or 2's continue bracketing rule by our inductive pf in sl_2 case.

QED.

(F_2 similar)

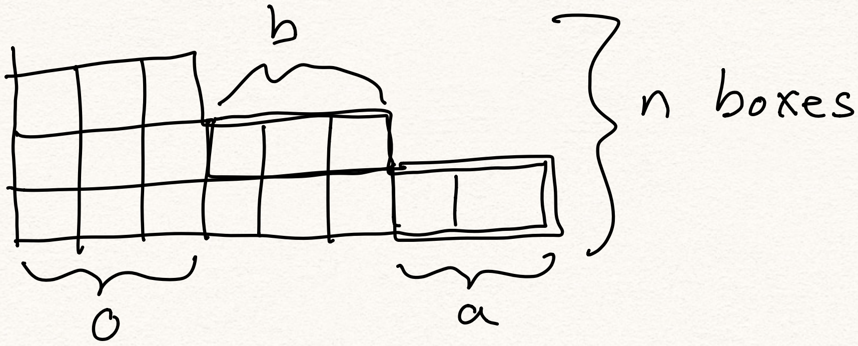
$$\boxed{1} \oplus \begin{tikzcd} \boxed{2} \arrow[d] \\ \boxed{3} \end{tikzcd}$$

Corollary: A word is highest weight (killed by raising operators $E_1 = E_{12}$, $E_2 := E_{23}$) iff its 1,2-subword is ballot (every suffix contains at least as many 1's as 2's) and its 2,3-subword is ballot.

(every suffix contains at least as many 2's as 3's)

Note: One irred. component for each highest weight elt.

Thm: Number of times $V_{(a,b)}$ occurs in $V_{(1,0)}^{\otimes n}$ is # SYT's of shape:



Ex: $V_{(1,0)}^{\otimes 3}$:

111
$\downarrow F_1$
112

121

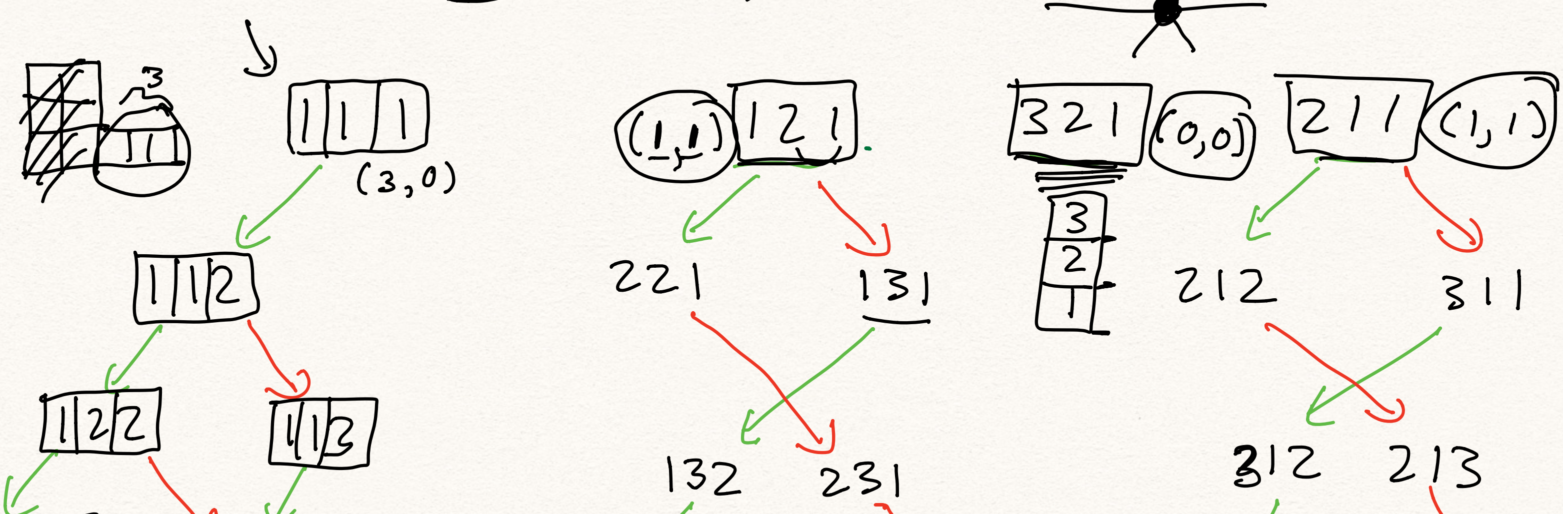
⋮

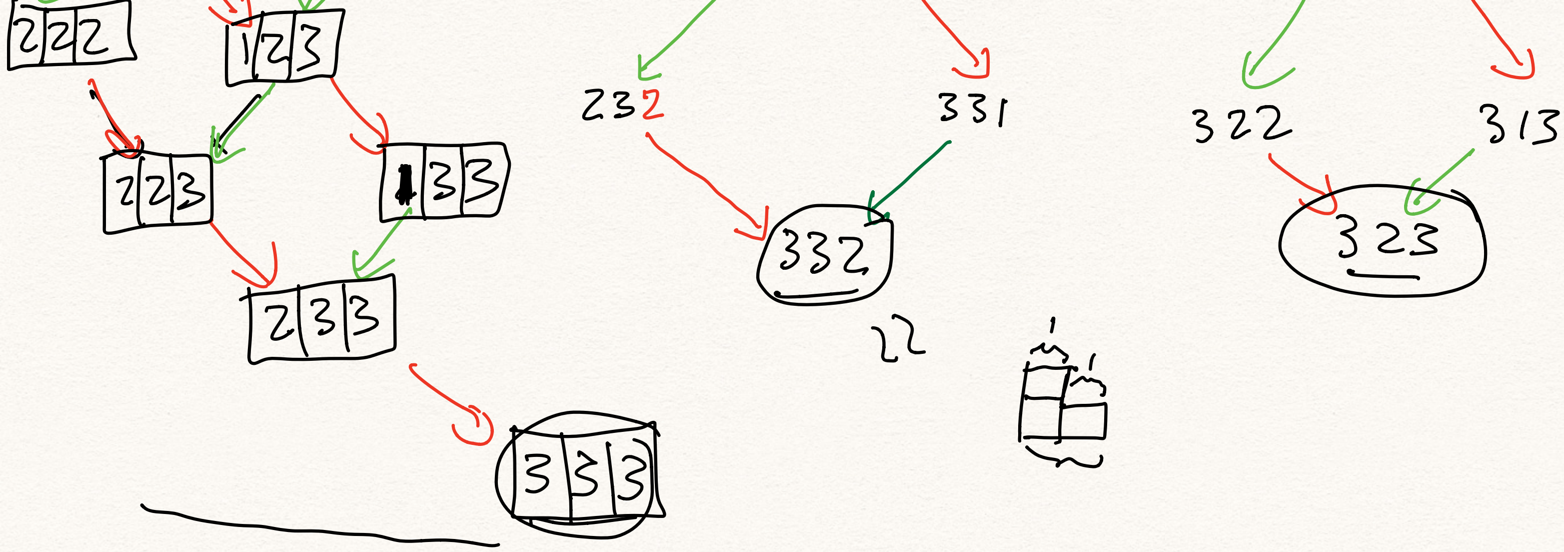
Shortcut:
Find all h.w. words first. (ballot: as read R to L, at least as many i's as i+1's)

H.W. words: $\boxed{111, \ 121, \ 321, \ 211}$

Length 4 hw words: 1111, 2121, 3121
3211, 1121, 1211, 2111, 2211

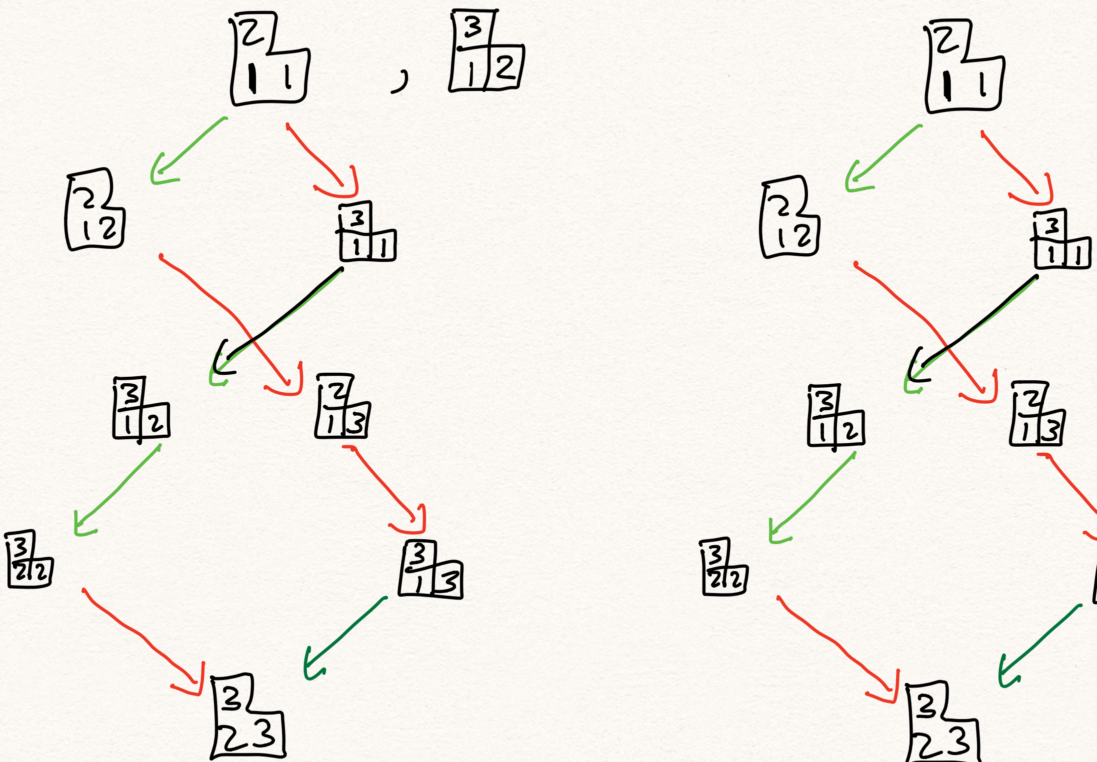
1321





$$\boxed{V_{(1,0)}^{\otimes 3} = V_{(3,0)} \oplus 2V_{(1,1)} \oplus V_{(0,0)}.}$$

Apply RSK to these two diagrams: (words 121, 211)



Rec tab: $\begin{array}{|c|c|}\hline 3 & \\ \hline 1 & 2 \\ \hline \end{array}$

Rec: $\begin{array}{|c|c|}\hline 2 & \\ \hline 1 & 3 \\ \hline \end{array}$

$\Rightarrow$ # copies is # possible recording tableaux on this shape, which equals #SYT of this shape.

<u>Claim:</u> Insertion tableau of any highest weight word is $\begin{array}{|llllll} 3 & 3 & & & & \\ 2 & 2 & 2 & & & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{array} \in$

<u>Steps:</u> (1) Knuth equivalence classes $\Leftrightarrow$ insertion tableau.

(2) Knuth equivalence moves don't change word of unbracketed 1's, 2's or 2's, 3's.

(3) Reading word of $\begin{array}{|lllll} 3 & 3 & & & \\ 2 & 2 & 2 & & \\ 1 & 1 & 1 & 1 & 1 \end{array}$ is highest weight (ballot) $\boxed{33222111111}$

(1) <u>Lemma:</u> Two words are Knuth equiv iff they have the same insertion tableau.

<u>Proof:</u> Recall a <u>simple Knuth move</u> is one of:

(cons. subwords) $\quad bac \leftrightarrow bca \quad$ if $a < b \leq c$

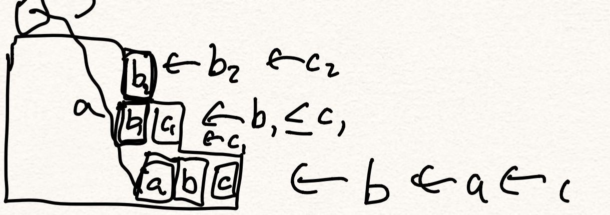
OR $\quad acb \leftrightarrow cab \quad$ if $a \leq b < c$

Two words w, w' are <u>Knuth equiv</u> if we can form w' from a sequence of Knuth moves starting at w.

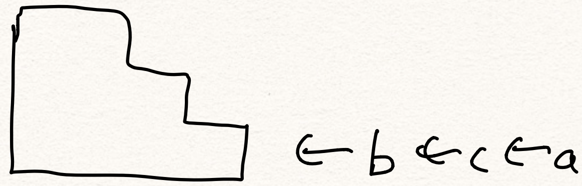
($\Rightarrow$) Suppose w, w' differ by a Knuth move.

<u>Case 1:</u> $\underbrace{w_1 \cdots w_k bac \sim}_{w} \to \underbrace{w_1 \cdots w_k bca \sim}_{w'}$.

Inserting b:



(ins $w_1 \cdots w_n$)



(ins $w_1 \cdots w_n$

<u>Recall (502)</u>: Insertion path goes up and weakly left.

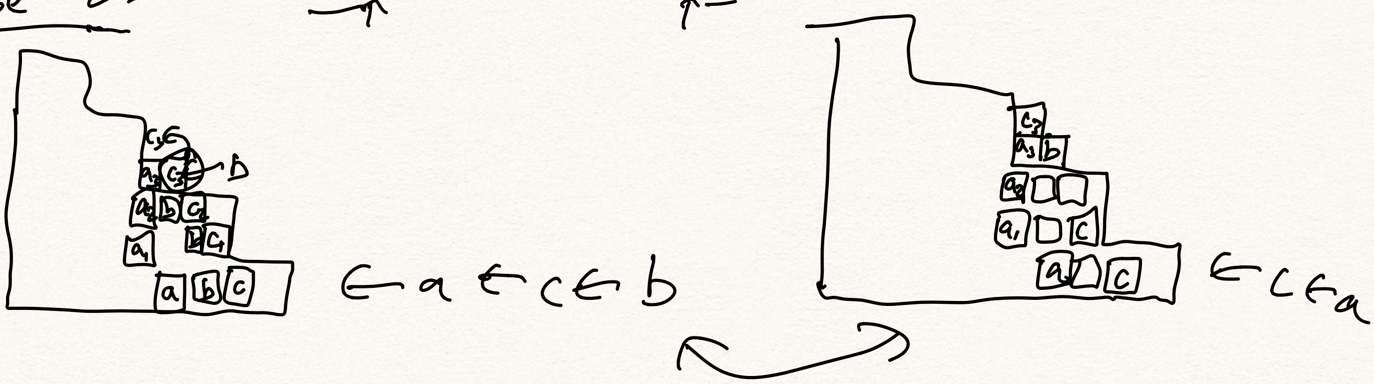
$a \le b$: Insertion path of a is weakly left of that of b.

$b < c$: Insertion path of c (after a) is <u>strictly</u> to right of b's path.

In w': c's path still strictly right of b's, a's is weakly left of b's

so $ins(w) = ins(w')$.

<u>Case 2:</u> $acb \leftrightarrow cab$



<u>Insert b:</u> path is strictly right of a's, weakly left of c's.

($\Leftarrow$) Want to show: If <u>$ins(w) = ins(v)$</u> $= T$ then <u>$w \sim v$</u>

Suffices to show they are both Knuth equivalent to the <u>reading word</u> of T

(Note: <u>$\text{ins}(\text{rw}(T)) = T$</u>)
(not hard, possibly 502)

Want: if $\text{ins}(w) = T$ then $w \sim \text{rw}(T)$.

<u>Claim:</u> $\text{rw}(T') \cdot x \sim \text{rw}(T' \leftarrow x)$ (this suffices by inducting on length of w)

concatenation; letter; inserting x into T'

$w = w_1 w_2 \cdots w_n$ $\quad \text{ins}(w) = \left(\boxed{w_1} \leftarrow w_2\right) \leftarrow w_3 \leftarrow \cdots$

<u>Pf of claim:</u> (by example)

$$T' = \begin{array}{l} 3\,4 \\ 2\,2\,3 \\ 1\,1\,1\,2\,2\,3\,5 \end{array} \qquad x = 1$$

$T' \leftarrow x$:

$$\begin{array}{l} \boxed{} \\ 3\,\boxed{4} \\ 2\,2\,\boxed{3} \\ 1\,1\,1\,\boxed{2}\,2\,3\,5 \leftarrow 1 \end{array} \rightsquigarrow \begin{array}{l} 4 \\ 3\,3 \\ 2\,2\,2 \\ 1\,1\,1\,1\,2\,3\,5 \end{array} \qquad T = T' \leftarrow x$$

Compare:

$\text{rw}(T') \cdot x$ vs $\text{rw}(T)$

$= 3422311122351$ $\qquad$ $433222\underbrace{1111235}$

$\overset{2}{\sim}$

3422311122315 $\qquad$ 4332221111235

$\overset{2}{\sim}$

3422311122135

$\overset{2}{\sim}$

$342231 1 1\boxed{2}1235$

$\wr$

3422311211235

$\wr$

3422312111235

$\wr$

$34223\,2\,\underbrace{1111235}$

$\wr$

3423221111235

$\wr$

3432221111235

$\wr$

4332221111235

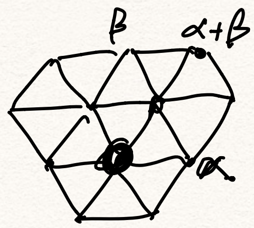
QED

Characters and Schur functions

Def: The character of a rep V of Lie alg $\mathfrak{g}$ (where $V = \underbrace{\bigoplus_{\alpha \in S} c_\alpha V_\alpha}_{\text{weight spaces}}$) is $\sum_{\alpha \in \Lambda} c_\alpha x^\alpha$

where $\boxed{x^\alpha}$ is a formal symbol satisfying

$$\boxed{x^\alpha x^\beta = x^{\alpha+\beta}}$$



Ex: chars of $\mathfrak{sl}_3$ reps: $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} =: x^\alpha$

where $\alpha = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3$

$$L_1 + L_2 + L_3 = 0$$

Polynomials in x_1, x_2, x_3 mod relation

$$\boxed{x_1 x_2 x_3 = 1}$$

$$x_1^4 x_2^2 x_3 = x_1^3 x_2^1$$

$V^{(a,b)}$: irr. rep. w/ highest wt (a,b)

V_α : weight space for α in rep V.

- $V^{(a,b)}_{(a,b)}$

$$\begin{tikzcd} & \alpha_0 \arrow[dl] \arrow[dr] & \\ \alpha_1 \arrow[dr] & & \alpha_2 \arrow[dl] \\ & \alpha_3 & \end{tikzcd} \quad V^{\alpha_0}$$

$$\underline{V^{\alpha_0} = V_{\alpha_0} \oplus V_{\alpha_1} \oplus V_{\alpha_2} \oplus V_{\alpha_3}}$$

Ex:

$$\begin{tikzcd} & \boxed{1\,1} \arrow[d, "F_1"'] & \\ & 12 \arrow[dl, "F_1"'] \arrow[dr, "F_2"] & \\ 22 \arrow[dr, "F_2"'] & & 13 \arrow[dl] \\ & 23 \arrow[d] & \\ & 33 & \end{tikzcd}$$

char.

$x_1^2 \qquad x_1^2 x_2^0 x_3^0$

$x_1 x_2$

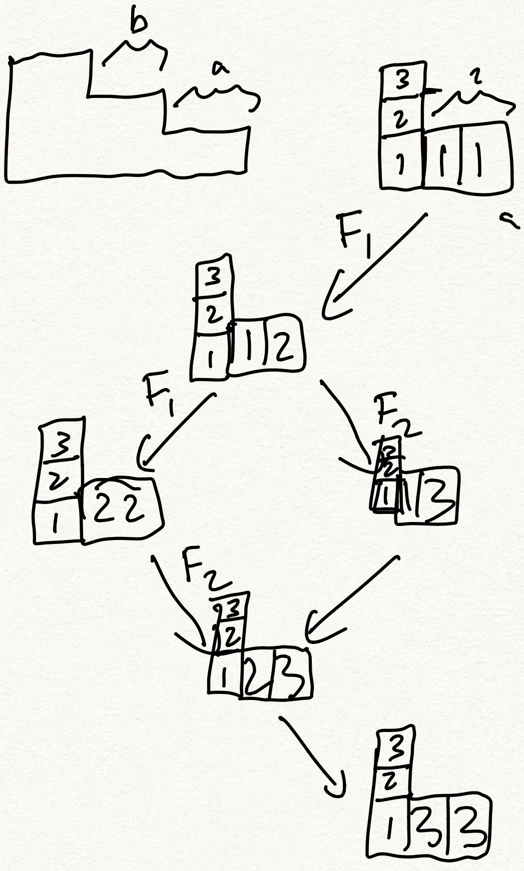
$x_2^2 \,,\; x_1 x_3$

$x_2 x_3$

x_3^2

$x_1^{\#1\text{'s}} \; x_2^{\#2\text{'s}} \; x_3^{\#3\text{'s}}$.

$$\rightsquigarrow \mathrm{ch}(V^{(2,0)}) = x_1^2 + x_1x_2 + x_2^2 + x_1x_3 + x_2x_3 + x_3^2 = s_{(2,0)}(x_1, x_2, x_3)$$



$V^{(2,0)}$

char:

$$x_1^3 x_2 x_3 + x_1^2 x_2^2 x_3 + \cdots$$

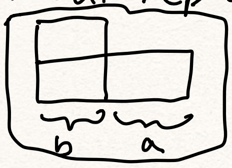
$$s_{(3,1,1)}(x_1,x_2,x_3) = \underbrace{x_1x_2x_3}\left(\underline{s_{(2,0)}}(x_1,x_2,x_3)\right)$$

α's are joint eigenvalues of $\mathfrak{h}$ = diag. matrices in $\mathfrak{g}$.

(trace = sum of eigenvalues)

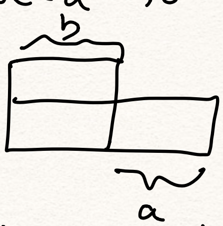
<u>Lemma:</u> $ch(V^{(a,b)}) \equiv s_{\underline{(a,b)}}(x_1,x_2,x_3) \pmod{x_1x_2x_3 = 1}$

irred. rep of h.w. (a,b)



<u>Pf:</u> Recall $s_\lambda(x_1,\dots,x_n) = \sum\limits_{\substack{\text{SSYT } T \\ \text{of shape } \lambda \\ \text{using only} \\ \text{letters } 1,\dots,n.}} x_1^{\#1\text{'s}} x_2^{\#2\text{'s}} \cdots x_n^{\#n\text{'s}}$

Need to show <u>every</u> SSYT of shape



w/ 1's, 2's, 3's can be obtained by sequence of F_1's, F_2's applied to $\begin{array}{|c|c|cc|}\hline 2 & 2 & & \\ \hline 1 & 1 & 1 & 1 \\ \hline\end{array}$.

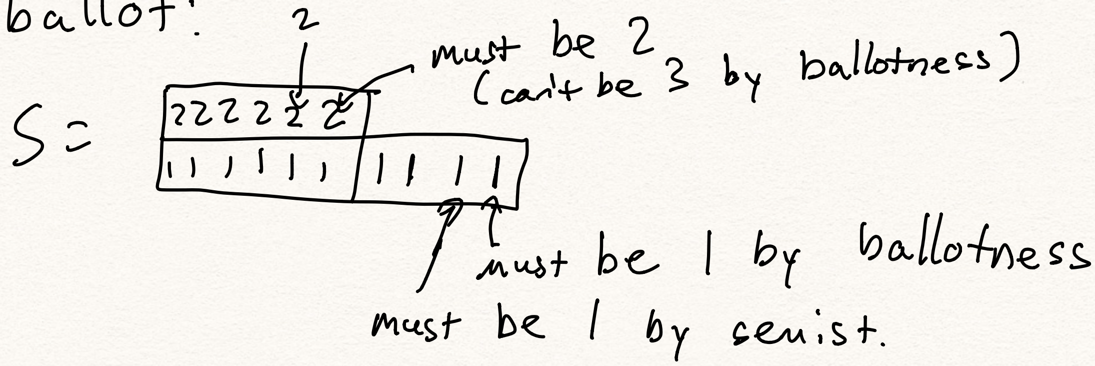
Given an SSYT T, if it's not h.w, we can apply E_1 or E_2 until get to a highest weight tab.

E_1: Bracket 1's, 2's, change 2 to 1

E_2: Bracket 2's, 3's, change 3 to 2.

get to a tab. S whose reading word

is ballot:



Reverse this process to get from S to T w/ F_1, F_2. ✓