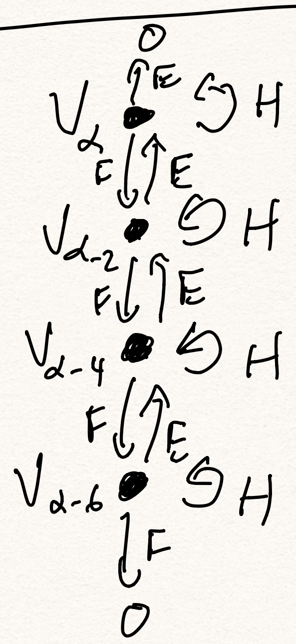
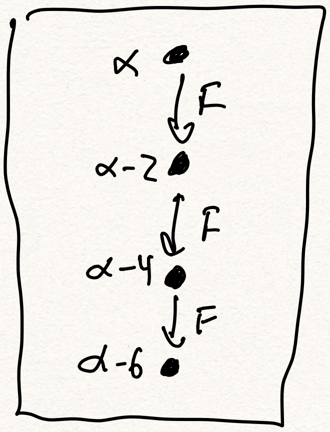
Reps. of sl_2 continued



$\leftarrow$ Irred. finite-dim'l rep of sl_2

Simplified:



<u>Thm:</u> For an irred. sl_2-rep $V_\alpha \oplus V_{\alpha-2} \oplus \cdots \oplus V_{\alpha-2n}$

α is a nonneg. integer and

$\alpha - 2n = -\alpha$

<u>Pf:</u> Let $v_\alpha \in V_\alpha$, let m smallest s.t. $(m=n+1)$

$$F^m v_\alpha = 0$$

Then
$$\begin{aligned} 0 &= \underline{E(F^m v_\alpha)} \\ &= EF(F^{m-1} v_\alpha) \\ &= ([E,F] + FE)(F^{m-1} v_\alpha) \\ &= (H + FE) F^{m-1} v_\alpha \end{aligned}$$

$$= H(F^{m-1}v_\alpha) + FEF^{m-1}v_\alpha$$

$$= (\alpha - 2m + 2)F^{m-1}v_\alpha + F\underbrace{EF}F^{m-1}v_\alpha$$

$$= (\alpha - 2m + 2)F^{m-1}v_\alpha + F([E,F] + FE)F^{m-2}v_\alpha$$

$$= (\alpha - 2m + 2)F^{m-1}v_\alpha + (\alpha - 2m + 4)F^{m-1}v_\alpha +$$

$$\underbrace{F^2EF^{m-2}v_\alpha}$$

$$\vdots$$

$$= \left(\underline{(\alpha - 2m + 2) + (\alpha - 2m + 4) + \cdots + \alpha}\right)F^{m-1}v_\alpha$$

$$= \underline{(\alpha - m + 1)\,m\,F^{m-1}v_\alpha}$$

$m \neq 0$, $F^{m-1}v_\alpha \neq 0$ so

$\alpha - m + 1 = 0$

$$\boxed{\alpha = m - 1}$$

Ex:

$$\begin{tikzcd}[row sep=small]
\bullet\ 5 \arrow[d] \\
\bullet\ 3 \arrow[d] \\
\bullet\ 1 \arrow[d] \\
\bullet\ -1 \arrow[d] \\
\bullet\ -3 \arrow[d] \\
\bullet\ -5
\end{tikzcd}$$

QED.

Cor: There is one irreducible $sl_2(\mathbb{C})$ representation, V_n, for every nonnegative integer n. (and there are no others).

Q: What do tensor products look like?

Tensor products of Lie alg. reps.

Recall: for $G \curvearrowright V$, $G \curvearrowright W$,

$G \curvearrowright V \otimes W$ by (groups, Lie groups)

$$\boxed{g(v \otimes w) = gv \otimes gw}$$

(Recall: $\otimes$ is bilinear)

ε method:

$X \in \mathfrak{g} = T_e G \iff$ "$I + \varepsilon X \in G$"

$$(I+\varepsilon X)(v \otimes w) = ((I+\varepsilon X)v) \otimes ((I+\varepsilon X)w)$$

$$= v \otimes w + (\varepsilon X v) \otimes w + v \otimes (\varepsilon X w) + \cancel{\varepsilon X v \otimes \varepsilon X w}$$

$$\cancel{\varepsilon^2 (Xv \otimes Xw)}$$

$$= I(v \otimes w) + \varepsilon \underline{(Xv \otimes w + v \otimes Xw)}$$

Conclusion: If V, W are reps. of Lie alg $\mathfrak{g}$, $X \in \mathfrak{g}$, then $\mathfrak{g}$ acts on $V \otimes W$ by

$$\boxed{X(v \otimes w) = \underline{Xv} \otimes w + v \otimes \underline{Xw}}$$

$\rho : \mathfrak{g} \to \mathfrak{gl}(V)$
$\sigma : \mathfrak{g} \to \mathfrak{gl}(W)$

$?? : \mathfrak{g} \to \mathfrak{gl}(V \otimes W)$

Ex: What is

$$\begin{tikzcd} & 0 & & & 0 & \\ (1) & v_1 \arrow[u] \arrow[d, shift right] \arrow[loop right, "H"] & & & w_1 \arrow[u] \arrow[d, shift right] & (1) \\ & & & \otimes & & \\ (-1) & v_2 \arrow[u, shift right] \arrow[d] \arrow[loop right, "H"] & & & w_2 \arrow[u, shift right] \arrow[d] & (-1) \\ & 0 & & & 0 & \end{tikzcd}$$

Basis:

$v_1 \otimes w_1$, $v_2 \otimes w_1$,

$v_1 \otimes w_2$, $v_2 \otimes w_2$

$$\boxed{\begin{tikzcd} 0 \\ v_1 \otimes w_1 \arrow[u, "E"] \arrow[d, "F"] \arrow[loop right, "H\ (2)"] \\ \underline{v_2 \otimes w_1} + \boxed{\underline{v_1 \otimes w_2}} \arrow[d, "F"] \arrow[loop right, "H\ (0)"] \\ 2 v_2 \otimes w_2 \arrow[d, "F"] \arrow[loop right, "H\ (-2)"] \\ 0 \end{tikzcd}}$$

$H(v_1 \otimes w_1) = \underline{Hv_1} \otimes w_1 + v_1 \otimes \underline{Hw_1}$

$= 2 v_1 \otimes w_1$

$F(v_1 \otimes w_1)$

$F(v_2 \otimes w_1) = \cancel{Fv_2 \otimes w_1} + v_2 \otimes Fw_1$

$= v_2 \otimes w_2$

Another irred. component:

$$\bullet \circlearrowleft H \quad (0)$$

$$v_2 \otimes w_1 - v_1 \otimes w_2$$

Summary:

$$\begin{array}{c} 1 \bullet \\ \downarrow \\ \underline{-1} \bullet \end{array} \otimes \begin{array}{c} 1 \bullet \\ \downarrow \\ \underline{-1} \bullet \end{array} = \begin{array}{c} 2 \bullet \\ | \\ 0 \bullet \\ | \\ -2 \bullet \end{array} \oplus \bullet\ 0$$

i.e.

$$\boxed{V_1 \otimes V_1 = V_2 \oplus V_0}$$

Corresp. $\begin{pmatrix}\text{reps of Lie gp}\\ G\end{pmatrix} \longleftrightarrow \begin{pmatrix}\text{reps. of Lie}\\ \text{alg } \mathfrak{g} = T_e G\end{pmatrix}$

$$\rho : G \to GL(V) \longleftrightarrow d\rho : \mathfrak{g} \to \underline{\mathfrak{gl}(V)}$$

Because $\mathfrak{gl}(V) = T_e(GL(V))$

same V for Lie gp, Lie alg. reps.