Math 601: Advanced Combinatorics I
Homework 9 - Due Nov 20

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (4 points) Let V be an n-dimensional vector space. Recall that $\Lambda^3(V)$ is defined as the quotient of $V \otimes V \otimes V$ by the subspace generated by the 'antisymmetry relations' $v_1 \otimes v_2 \otimes v_3 - \mathrm{sgn}(\pi) v_{\pi(1)} \otimes v_{\pi(2)} \otimes v_{\pi(3)}$. Verify directly that, as a $\mathrm{GL}(V)$-representation, the representation $\Lambda^3(V)$ is isomorphic to the subspace of $V^{\otimes 3}$ generated by all vectors of the form

   $$v_1 \otimes v_2 \otimes v_3 - v_2 \otimes v_1 \otimes v_3 - v_3 \otimes v_2 \otimes v_1 - v_1 \otimes v_3 \otimes v_2 + v_2 \otimes v_3 \otimes v_1 + v_3 \otimes v_1 \otimes v_2.$$

   Explain why the latter construction coincides with the Schur functor construction $S_{(1,1,1)}(V)$.

2. (3 points) For the standard tableau T given by

   $$\begin{array}{|c|c|} \hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

   compute the Young symmetrizer $\varepsilon_T \in \mathbb{C}[S_4]$, and apply it to the vector $v_1 \otimes v_2 \otimes v_3 \otimes v_4$.

3. (3 points) Let V be an m-dimensional vector space with basis $\{e_1, e_2, e_3, \ldots, e_m\}$. Write down the vector corresponding to the tableau

   $$\begin{array}{|c|c|} \hline 2 & 1 \\ \hline 1 & 2 \\ \hline \end{array}$$

   in $S_{(2,2)}(V)$. How can this be expressed in terms of the basis vector corresponding to the semistandard tableau

   $$\begin{array}{|c|c|} \hline 2 & 2 \\ \hline 1 & 1 \\ \hline \end{array}$$

   (obtained by flipping the right-hand column upside down) in $S_{(2,2)}(V)$?

4. (4 points) Recall that a **column exchange** in a (not necessarily semistandard) tableau consists of choosing two columns, a subset Y of the first and a subset X of the second for which $|X| = |Y|$, and interchanging the elements of X and Y, maintaining the relative order of each.

   Recall also that in $S_\lambda(V)$, an element represented by a tableau is equivalent to the sum of the tableaux formed by choosing two rows and a subset X of the second row and exchanging it with some subset Y of the first (where the sum ranges over all possible such Y). Finally, recall that switching two elements in a column negates the tableau.

   Apply a series of these column exchange relations to "straighten" the tableau

   $$\begin{array}{|c|c|c} \hline 4 & 4 & \multicolumn{1}{c}{} \\ \hline 3 & 3 & \multicolumn{1}{c|}{2} \\ \hline 2 & 1 & \multicolumn{1}{c|}{1} \\ \hline \end{array},$$

   thereby expressing the element

   $$(e_1 \wedge e_3 \wedge e_4) \otimes (e_2 \wedge e_3 \wedge e_4) \otimes (e_1 \wedge e_2)$$

   in terms of basis elements corresponding to semistandard tableaux in $S_{(3,3,2)}(V)$.