Math 601: Advanced Combinatorics I
Homework 8 - Due Nov 13

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (2 points) Is the word 132312 reduced? If so, compute the corresponding pair of tableaux under the Edelman-Greene bijection. If not, explain why.

2. (2 points) Is the word 321231 reduced? If so, compute the corresponding pair of tableaux under the Edelman-Greene bijection. If not, explain why.

3. (3 points) Use Edelman-Greene insertion to compute the number of reduced words in S_n of length $\binom{n}{2} - 1$. How many distinct elements of S_n have this length? How many reduced words does each such element have?

4. (4 points) Recall that an *inversion* of a permutation $p = p_1, p_2, \ldots, p_n$ written in list notation is a pair of indices $i < j$ such that $p_i > p_j$. We write $\mathrm{inv}(p)$ for the number of inversions of p. Show that $\mathrm{inv}(p)$ is equal to the length $\ell(p)$ of any reduced word for p.

   HINTS:

   (a) First show that $\ell(p) \leq \mathrm{inv}(p)$ by induction on n.

   (b) Then show that $\mathrm{inv}(p) \leq \ell(p)$ by fixing n and inducting on the length of the reduced word.

5. (5 points) For a **signed** permutation $p = p_1, p_2, \ldots, p_n$ written in list notation, define

$$\mathrm{inv}_B(p) = \mathrm{inv}(p) + \mathrm{neg}(p) + \mathrm{nsp}(p)$$

   where:

   - $\mathrm{inv}(p)$ is the number of pairs $i < j$ such that $p_i > p_j$,
   - $\mathrm{neg}(p)$ is the number of negative entries in p,
   - $\mathrm{nsp}(p)$ is the number of pairs $\{p_i, p_j\}$ such that $p_i + p_j < 0$ ("negative sum pairs").

   Prove that $\mathrm{inv}_B(p)$ is the length of any reduced word for p in the hyperoctahedral group H_n.