Math 601: Advanced Combinatorics I
Homework 7 - Due Nov 6

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. Let $\mathcal{C}_2$ be be the standard crystal for type C_2.
   (a) (3 points) Draw the tensor product of crystals $\mathcal{C}_2 \otimes \mathcal{C}_2$. How many connected components are there, and what are their highest weights?
   (b) (3 points) Derive and prove a rule for when a word in the letters $1, 2, \overline{2}, \overline{1}$ is highest weight in $\mathcal{C}_2^{\otimes n}$. Use your formula to compute how many connected components there are in $\mathcal{C}_2 \otimes \mathcal{C}_2 \otimes \mathcal{C}_2$.
2. (5 points) Let $\mathcal{B}_2$ be the standard crystal for type B_2. Derive and prove a rule for when a word in the letters $1, 2, 0, \overline{2}, \overline{1}$ is a highest weight word in $\mathcal{B}_2^{\otimes n}$. Use your formula to compute the decomposition of the representation corresponding to $\mathcal{B}_2 \otimes \mathcal{B}_2 \otimes \mathcal{B}_2$ into irreducibles.
3. (2 points) Show that the transpositions $s_i = (i \;\; i+1)$ satisfy the braid relations $s_i^2 = 1$ and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for all i.
4. (3 points) Show that the longest word $s_1 s_2 s_1$ in S_3, when applied to $\mathfrak{sl}_3$ crystals, gives the "up-down symmetry" of the crystal diagrams (in other words, the highest weight element gets sent to the lowest weight, and if an element x can be obtained from the highest weight by a sequence of f_i operators, then its image under $s_1 s_2 s_1$ can be obtained from the lowest weight by the corresponding sequence of e_i operators).
5. (3 points) What is the size of the Weyl group of type G_2? Write out its elements as reduced words in the two simple reflections s_1, s_2 corresponding to the two simple roots of G_2.