Math 601: Advanced Combinatorics I
Homework 6 - Due Oct 16

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. In this problem we will construct a bijection between the two different sets of combinatorial objects that we have shown are counted by the Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$. *If you only do some parts of the problem, you may use other parts as facts even if you haven't written up those parts.*

   We define $P^{\nu}_{\lambda\mu}$ to be the set of all pairs of semistandard Young tableaux (S,T) of shapes λ and μ whose concatenated reading word is ballot of content ν. We also define $Q^{\nu}_{\lambda\mu}$ to be the set of all skew semistandard tableaux with of shape ν/λ and content μ whose reading word is ballot. We will construct a bijection from $P^{\nu}_{\lambda\mu}$ to $Q^{\nu}_{\lambda\mu}$.

   (a) (3 points) Show that if two words w and v are Knuth equivalent, then removing the smallest k letters from each of w and v (with ties broken in order from left to right) results in two Knuth equivalent words w' and v'. (Hint: It suffices to show it for just one letter removed, and it also suffices to assume that w and v differ by an elementary Knuth move.)

   (b) (2 points) Given a pair (S,T) in $P^{\nu}_{\lambda\mu}$, show that T must be the unique highest weight tableau of shape μ. Conclude that the pair (T,T) corresponds under RSK to a two line array

   $$\begin{pmatrix} a_1 & a_2 & \cdots & a_{|\mu|} \\ b_1 & b_2 & \cdots & b_{|\mu|} \end{pmatrix}$$

   where $a_1, a_2, \ldots, a_{|\mu|}$ is the unique weakly increasing word of content μ and $b_1, b_2, \ldots, b_{|\mu|}$ is a ballot word.

   (c) Insert $b_1, ..., b_n$ into S and write $a_1, \ldots, a_{|\mu|}$ into the boxes that appear in order. Let R be the resulting skew tableau with letters $a_1, \ldots, a_{\mu}$.

      i. (2 points) Show that R shape ν/λ and content μ.

      ii. (4 points) To show it is ballot, consider the tableau $\tilde{R}$ formed by filling the shape λ underneath R with a highest weight word of negative entries $0, -1, -2, \ldots$, in other words, fill the top row of λ with 0's, the next with -1, the next with -2, and so on. Let V be the highest weight tableau of shape ν. Apply RSK to the pair $(\widetilde{R}, V)$, and use part (a) of this problem to show that the letters corresponding to the entries of $\widetilde{R}$ in the resulting two-line array form a highest weight word. Conclude that the reading word of R is ballot.

      iii. (1 point) Conclude that the map $(S,T) \mapsto R$ using the above construction is a well-defined function from $P^{\nu}_{\lambda\mu}$ to $Q^{\nu}_{\lambda\mu}$.

   (d) (5 points) Explain why this map is a bijection.

2. (3 points) Suppose V is a representation of $\mathfrak{sl}_n$, and let v_α be a weight vector of weight $\alpha \in \mathfrak{h}^*$. Show that $E_{i,i+1}v_\alpha$ is a weight vector of weight $\alpha + \alpha_i$ where α_i is the simple root $L_i - L_{i+1}$. (Hint: The proof should be very similar to how we showed in class that the weights differ by 2 in representations of $\mathfrak{sl}_2$, or that the weights differ by a root in representations of $\mathfrak{sl}_3$.)

3. (4 points) Consider a word of 1's and 2's. Show that if we remove a 2 from this word, then at most one 1 that was bracketed becomes unbracketed, and all other 1's retain their status (of being bracketed with some 2 or not).