Math 601: Advanced Combinatorics I
Homework 5 - Due Oct 9

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. Recall that the *complete homogeneous symmetric function* $h_\mu(x_1, \dots, x_n)$, for a partition $\mu = (\mu_1, \dots, \mu_k)$, can be defined as the product $h_{\mu_1} \cdots h_{\mu_k}$ where h_d is the sum of all monomials in $x_1, \dots, x_n$ of degree d.

   (a) (1 point) Which Schur function in n variables is $h_d(x_1, \dots, x_n)$ equal to?

   (b) (2 points) Show that $h_\mu(x_1, x_2, x_3)$ is the character of the tensor product of the k irreducible $\mathfrak{sl}_3$ representations
   $$V^{(\mu_1,0)} \otimes \cdots \otimes V^{(\mu_k,0)}$$
   (where $V^{(a,b)}$ denotes the irreducible representation with highest weight (a, b)).

   (c) (5 points) Consider each $V^{(\mu_i,0)}$ in the tensor product of the previous problem as the tableau crystal on the one-row shape μ_i. Then any node of the tensor product crystal corresponds to a choice of one-row tableaux $T_1, T_2, \dots, T_k$ where each T_i is a semistandard filling of the row μ_i (i.e. a weakly increasing sequence of letters $(T_i)_1, \dots, (T_i)_{\mu_i}$). This may be represented as a two-line array
   $$\begin{pmatrix} 1 & 1 & \cdots & 1 & 2 & 2 & \cdots & 2 & \cdots & k & k & \cdots & k \\ (T_1)_1 & (T_1)_2 & \cdots & (T_1)_{\mu_1} & (T_2)_1 & (T_2)_2 & \cdots & (T_2)_{\mu_2} & \cdots & (T_k)_1 & (T_k)_2 & \cdots & (T_k)_{\mu_k} \end{pmatrix}$$
   Use RSK on these two-line arrays to show that the number of times $V^\lambda := V^{(\lambda_1-\lambda_2, \lambda_2-\lambda_3)}$ appears in this tensor product is equal to the number of semistandard Young tableaux of shape λ and content μ.

   (d) (1 point) Recall from Math 502 the definition of the Kostka numbers $K_{\lambda\mu}$ and state the definition here.

   (e) (1 point) Explain how we could have deduced part (c) immediately using the formula
   $$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda$$
   (or alternatively, derived this formula from part (c)). What happens to this formula when we only restrict this identity to three variables x_1, x_2, x_3?

2. Recall that the *elementary symmetric function* $e_\mu(x_1, \dots, x_n)$, for a partition $\mu = (\mu_1, \dots, \mu_k)$, can be defined as the product $e_{\mu_1} \cdots e_{\mu_k}$ where e_d is the sum of all square-free monomials in $x_1, \dots, x_n$ of degree d. Note that if $d > n$ then $e_d(x_1, \dots, x_n) = 0$.

   (a) (2 points) Recall the operator ω on symmetric functions and apply it to both sides of the formula
   $$h_\mu = \sum_\lambda K_{\lambda\mu} s_\lambda.$$
   Explain what happens when we restrict both sides to the three variables x_1, x_2, x_3.

   (b) (3 points) Interpret your answer to part (a) in terms of expressing a tensor product of irreducible representations of $\mathfrak{sl}_3$ as a direct sum of irreducibles.

3. (3 points) Explain why there is no $\mathfrak{sl}_3$ representation whose character is equal to the *power sum symmetric function* $p_{(2)}(x_1, x_2, x_3)$.