Math 601: Advanced Combinatorics I
Homework 4 - Due Oct 2

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

NOTE: Some problems below concern $\mathfrak{sl}_2$ and others concern $\mathfrak{sl}_3$. Read carefully!

1. (3 points) Describe a "ballot-type" condition for a word of 1's and 2's to be *lowest weight* for $\mathfrak{sl}_2$, that is, that F sends the word to 0. Prove that your condition is correct.

2. (3 points) In the $\mathfrak{sl}_3$ tableau crystals discussed in class, describe the *lowest weight* tableau of shape λ. Conclude that every irreducible $\mathfrak{sl}_3$ representation has a unique lowest weight.

3. (3 points) Write the adjoint representation of $\mathfrak{sl}_3$ as a direct sum of irreducible representations. Draw its weight spaces as dots on the triangular weight lattice. How does this correspond to root system of $\mathfrak{sl}_3$?

4. **Compatibility of Jeu de taquin and Knuth equivalence**: Recall that an *elementary Knuth move* is an operation on triples of consecutive letters in a word, of one of the following forms:

$$bac \leftrightarrow bca \qquad \text{where } b < a \leq c$$

or

$$acb \leftrightarrow cab \qquad \text{where } b \leq a < c$$

Two words are *Knuth equivalent* if they can be obtained from one another by a sequence of elementary Knuth moves.

   (a) (3 points) Let T be a skew semistandard Young tableau having only 1's and 2's as its entries, and let T' be the result of applying an inner Jeu de taquin slide to T. Show that the reading word of T is Knuth equivalent to that of T'.

   (b) (6 points) Let T be a skew semistandard Young tableau (of any content), and let T' be obtained as above. Show that the reading word of T' is Knuth equivalent to that of T.

5. (4 points) **Compatibility of Jeu de taquin and crystal operators ($\mathfrak{sl}_2$ case)** Let T be a skew semistandard Young tableau with only 1's and 2's as entries, and let T' be the result of applying an inner Jeu de taquin slide to T. Show that $F(T')$ is the result of applying an inner Jeu de taquin slide to $F(T)$.

6. (2 points) Compute $V_{0,1} \otimes V_{0,1}$ as a direct sum of irreducible representations of $\mathfrak{sl}_3$, by interpreting each in terms of two types of $\mathfrak{sl}_2$ chains and using the L method.

7. (2 points) Compute $V_{0,1} \otimes V_{0,1}$ as a direct sum of irreducible representations of $\mathfrak{sl}_3$ using the Littlewood-Richardson rule. What Schur function product does this correspond to, and with how many variables?