Math 601: Advanced Combinatorics I
Homework 3 - Due Sep. 18

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. **Writing $\mathfrak{sl}_2(\mathbb{C})$ representations in terms of matrices:**
   (a) (1 point) Draw the three-dimensional irreducible representation V_2 of $\mathfrak{sl}_2(\mathbb{C})$ as a graph with edges labeled by E, F, H.
   (b) (3 points) Consider the basis $\{v, u, w\}$ of V_2 given by three eigenvectors of H satisfying $Fv = u$ and $Fu = w$. Write the images of E, F, H in $\mathfrak{gl}(V_2)$ as 3×3 matrices with respect to this basis.
   (c) (1 point) If these 3×3 matrices are e, f, h, confirm that $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$.
2. (1 points) Let V_3 be the (four-dimensional) irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight 3. Write the tensor product $V_3 \otimes V_3$ as a direct sum of irreducible representations.
3. (2 points) Write the tensor product $V_2 \otimes V_2 \otimes V_2$ as a direct sum of irreducible representations.
4. (3 points) A **highest weight vector** of a representation of $\mathfrak{sl}_2$ is a vector v such that $Ev = 0$. Let v be a vector in $V_1^{\otimes n}$, written as a word w of 1's and 2's. Show that v is a highest weight vector if and only if the word $w = w_1 \cdots w_n$ has the *ballot* (also called the *lattice* or *Yamanouchi* property) that for all $i \leq n$, the suffix $w_i w_{i+1} \cdots w_n$ has at least as many 1's as 2's.
5. (5 points) Two words p, q of length n in letters 1 and 2 are *Knuth equivalent* to each other if and only if p can be obtained from q by a sequence of **elementary Knuth moves**, defined as follows. Given three consecutive letters of the form 212, it can be changed to 221 (or vice versa) in the middle of the word. Alternatively consecutive letters 121 can be changed to 211 and vice versa.

   Prove that two words in letters 1 and 2 are Knuth equivalent if and only if they have the same RSK insertion tableau.