Math 601: Advanced Combinatorics I
Homework 2 - Due Sep. 11

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (2 points) Show that the commutator bracket $[X, Y] = XY - YX$ is a Lie bracket, by showing that it is antisymmetric and satisfies the Jacobi identity

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

2. Using the formal infinitesimal ϵ satisfying $\epsilon^2 = 0$ as described in class on Friday, Sept. 4, describe the elements of the Lie algebra corresponding to each of the following Lie groups.
   (a) (1 point) $\mathrm{SO}_n(\mathbb{C})$
   (b) (1 point) $\mathrm{Sp}_{2n}(\mathbb{C})$
   (c) (1 point) The torus $T_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of diagonal invertible matrices
   (d) (1 point) The Borel subgroup $B_n(\mathbb{C}) \subseteq \mathrm{GL}_n(\mathbb{C})$ of upper triangular invertible matrices

3. (2 points) Show, using the ϵ method, that the Lie algebra of the orthogonal group $O_n(\mathbb{C})$ is isomorphic to that of the special orthogonal group $\mathrm{SO}_n(\mathbb{C})$. Why does this not contradict the bijective correspondence between connected Lie groups and Lie algebras?

4. (2 points) Compute the character of the identity representation of $\mathrm{GL}_n(\mathbb{C})$, that is, the identity map $\rho : \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$. Which Schur function is it?

5. (2 points) Prove that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$ for any two $n \times n$ matrices X and Y. Conclude that the bracket $[X, Y] = XY - YX$ is a well-defined Lie bracket on

$$\mathfrak{sl}_n(\mathbb{C}) = \{M \in \mathrm{Mat}_n(\mathbb{C}) : \mathrm{tr}(M) = 0\}.$$