Math 601: Advanced Combinatorics I
Homework 10 - Due Dec 16

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for details.

Problems

1. (3 points) Let T be the (nonstandard) tableau

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline \end{array}.$$

   Write the garnir polynomial f_T in terms of the two basis elements f_S and $f_{S'}$ of the Specht module $M_{(2,2)}$, where S and S' are the two standard Young tableaux

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array}, \qquad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}$$

   of shape $(2,2)$.

   If you do this using column exchange relations, verify that your equation holds either using Sage or by multiplying out the polynomials by hand.

2. (4 points) The Specht module $M_{(2,2)}$ is a 2-dimensional S_4 representation with basis elements f_S and $f_{S'}$ mentioned above. Consider the elements id, (12), (123), $(12)(34)$, and (1234) of S_4, each of which represents a different conjugacy class. Describe how each of these elements acts as a 2×2 matrix with respect to the basis $f_S, f_{S'}$, and compute each of their traces, giving the row in the character table corresponding to the partition $(2,2)$.

   Hint: Note that the permutations act on the tableau to give a different f_T, which then can be straightened as in the previous problem. For instance, $(12)f_S = f_{(12)S} = f_S - f'_S$ and $(12)f_{S'} = f_{(12)S'} = -f_{S'}$, so the matrix corresponding to (12) is

$$\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$$

   which has trace 0.

   Hint 2: If you want to check your work, apply the Frobenius map to your answer to get a symmetric function and use Sage to compute it to Schur functions. You should get the Schur function $s_{(2,2)}$.

3. (3 points) Prove that the number of partitions of n into odd parts equals the number of partitions of n into distinct parts.

4. (4 points) Let q_n (for $n \geq 0$) be the symmetric functions which we defined in class by the generating function identity

$$\sum_{n \geq 0} q_n t^n = \prod_{j=1}^{\infty} \frac{1 + x_j t}{1 - x_j t}.$$

   Also define $q_\lambda = q_{\lambda_1} \cdots q_{\lambda_k}$ for any partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Show that

$$\sum_{\lambda} q_\lambda(x_1, x_2, \ldots) m_\lambda(y_1, y_2, \ldots) = \prod_i \frac{1 + x_i y_i}{1 - x_i y_i}$$

   where the sum is over all partitions λ.