# Math 601: Advanced Combinatorics I
# Cumulative Homework C - Due Dec 4

For this homework, all problems must be handed in to receive full credit. The problems are worth a total of 30 points. No collaboration is allowed, and it should be treated as a mini-take-home midterm/review.

## Problems

1. (Short answer: 1 point each, 10 total) Compute the following. No explanations or proofs are necessary for this problem.

   (a) The matrix tensor product
   $$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \otimes \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$$

   (b) The number of ballot words of 1's and 2's of length 9.

   (c) The decomposition of the $\mathfrak{sl}_2$ representation $V^{(3)} \otimes V^{(5)}$ into irreducibles.

   (d) $\dim(V^{(2,1,0)})$, where $V^{(2,1,0)}$ is the irreducible representation of $\mathfrak{sl}_3$ with highest weight $(2,1,0)$ (modulo $(1,1,1)$). That is, as a tableau crystal, the highest weight element has two 1's and one 2.

   (e) The number of times the irreducible representation $V^{(2,1,0)}$ of $\mathfrak{sl}_3$ appears in the tensor product $V^{(1)} \otimes V^{(1)} \otimes V^{(1)}$.

   (f) The number of times $V^{(3,2,1)}$ appears in the tensor product $V^{(2,1)} \otimes V^{(2,1)}$ of $\mathfrak{sl}_3$ representations.

   (g) $F_2(12113323211231)$ where $F_2$ is the type A crystal lowering operator for the simple root $\alpha_2$.

   (h) The size of the Weyl group of type $B_6$.

   (i) $\dim(V_{(3,2,1)})$, where $V_{(3,2,1)}$ is the irreducible representation of $S_6$ corresponding to the partition $(3,2,1)$ under Schur-Weyl duality.

   (j) The number of elements in the conjugacy class of cycle type $(3,2,2,2,1)$ in $S_{10}$.

2. (5 points) Use the $\varepsilon$ method to show that the Lie algebra of $\mathrm{SL}_n(\mathbb{C})$, the group of matrices of determinant 1, is equal to $\mathfrak{sl}_n(\mathbb{C})$, the Lie algebra of matrices with trace 0.

3. (5 points) Show that the longest word $s_1s_2s_1$ in $S_3$, when applied to $\mathfrak{sl}_3$ crystals, gives the "up-down symmetry" of the crystal diagrams. In other words, show that the highest weight element gets sent to the lowest weight, and if an element $x$ can be obtained from the highest weight by a sequence of $f_i$ operators, then its image under $s_1s_2s_1$ can be obtained from the lowest weight by the corresponding sequence of $e_i$ operators.

4. (5 points) Let $R$ denote the *regular representation* of $S_3$ acting on the vector space of formal linear combinations of elements of $S_3$, where the action is by left multiplication. Compute the character $\chi_R$ by computing the trace of the action on each conjugacy class. Write the Frobenius character $\mathrm{Frob}(\chi_R)$ in the power sum basis, and then use Sage (or a computation by hand if you're brave) to convert it to the Schur basis. What does this say about the decomposition of $R$ into irreducible $S_3$ representations?

5. (5 points) Suppose $V$ is a representation of $S_4$ whose Frobenius image $\mathrm{Frob}(\chi_V)$ is $s_{(2,2)} + 3s_{(3,1)}$. Use Sage to convert this symmetric function to the power sum basis. From this expansion, deduce each value of the character $\chi_V(\mu)$ for every conjugacy class $\mu$ of $S_4$.