Math 601: Advanced Combinatorics I
Cumulative Homework B - Due Oct. 23

For this homework, all problems must be handed in to receive full credit. The problems are worth a total of 30 points. No collaboration is allowed, and it should be treated as a mini-take-home midterm/review.

Problems

1. (5 points total) Compute the decomposition into irreducibles of the tensor product of the $\mathfrak{sl}_2$ representations $V^{(3)}$ and $V^{(4)}$ in two ways:
   (a) (2 points) By using L-diagrams, and
   (b) (3 points) By interpreting them as tableau crystals on the one-row shapes (3) and (4) in the letters 1 and 2, and finding all pairs of reading words whose concatenation is ballot.

2. (10 points total) This problem concerns the adjoint representation of $\mathfrak{sl}_2$.
   (a) (2 points) What is the dimension of the adjoint representation of $\mathfrak{sl}_2$?
   (b) (2 points) What is its highest weight? Find a vector with that weight.
   (c) (4 points) Express the generators E, F, H of $\mathfrak{sl}_2$ as $d \times d$ matrices (where d is the dimension found in part (a) with respect to a basis of weight vectors for the representation. Explain why each of your matrices can be thought of as an element of $\mathfrak{gl}(\mathfrak{sl}_2)$.
   (d) (2 points) List the nonzero roots of $\mathfrak{sl}_2$.

3. (7 points) Prove that tableau crystals satisfy the Stembridge length axioms (S1) (see October 12-14 notes).

4. (8 points) Determine the graph structure of the $\mathfrak{sl}_3$ crystal having highest weight $(2, 0, 0)$ using only the Stembridge (and Kashiwara) axioms and without drawing any tableaux. Explain your reasoning at each step. Then verify that it matches what we get from the tableau crystal of one-row shape (2) on three letters $1, 2, 3$.

   (**Hints:** First draw a node x and label it with weight $(2, 0, 0)$. This will be the highest weight node, so $\varepsilon_1(x) = \varepsilon_2(x) = 0$ at that node. Then by Kashiwara axiom K2, we can deduce that $\varphi_1(x) = 2$ and $\varphi_2(x) = 0$. Thus, by axiom S0, $\varphi_1(x) = 2$ tells us that we can draw an arrow labeled f_1 out of x to a new node y, followed by another arrow labeled f_1 coming out of y to a new node z, which has no f_1 arrow coming out of it. Since x is the unique highest weight element, we also have $\varepsilon_2(y) = 0$. What do the length axioms S1 tell us about whether there is an f_2 arrow out of y or z? How can you continue from there?)