Math 601: Advanced Combinatorics I
Cumulative Homework A - Due Sep. 25

For this homework, all problems must be handed in to receive full credit. The problems are worth a total of 30 points. No collaboration is allowed, and it should be treated as a mini-take-home midterm/review.

Problems

1. (2 points) Consider the dihedral group D_4 (the symmetry group of a square, including reflections and rotations) acting on the plane by the corresponding reflection and rotation matrices. Show that this representation of D_4 is an irreducible 2-dimensional representation over both $\mathbb{R}$ and $\mathbb{C}$.

2. (7 points) Consider the action of the dihedral group D_4 on the vertices of a square, and assign each element of D_4 the corresponding 4×4 permutation matrix. Find the decomposition of this representation into irreducibles.

3. (2 points) Show that tensor product of vector spaces is symmetric:
$$V \otimes W \cong W \otimes V.$$

4. (3 points) Let $V_0 = \mathbb{C}$ be the trivial representation of the group G, in which every element of G acts as the identity (of size 1). Show that, for any representation W of G, we have $V_0 \otimes W \cong W$.

5. (3 points) Show that tensor product of vector spaces distributes over direct sum:
$$V \otimes (W \oplus U) \cong (V \otimes W) \oplus (V \otimes U).$$
(You do not need to prove right distributivity as well.)

6. (2 points) Use the ϵ method to find the Lie algebra associated to the Lie group B_n of upper triangular matrices in GL_n.

7. (a) (5 points) Show that the total number of ballot sequences of 1's and 2's of length $2n$ is $\binom{2n}{n}$, and that the number of ballot sequences of 1's and 2's of length $2n+1$ is $\binom{2n+1}{n+1}$.
   (b) (2 points) If V_1 is the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ having highest weight 1, what does this imply about the decompositions of $V_1^{\otimes 2n}$ and $V_1^{\otimes 2n+1}$ into irreducibles?

8. (2 points) Let V_i be the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$ with highest weight i. Compute the decomposition of $V_3 \otimes V_5$ into irreducibles.

9. (2 points) Starting with the ballot word 12211122121111, draw the corresponding $\mathfrak{sl}_2$ chain by applying the lowering operator F until it is no longer possible.