Math 566: Abstract Algebra I
Homework 13

10 points total. Due Friday, Dec 3 by 11:10 am in class. (In TWO WEEKS after this homework was posted. Happy Thanksgiving!)

Problems

1. (5 points - 1 point per part) Let R be a ring. For ideals $I, J \subset R$, define $I + J = \{a + b : a \in I, b \in J\}$, and define $I \cdot J = \{ab : a \in I, b \in J\}$.
   (a) Prove that $I + J$ is an ideal of R as well, and that it is the smallest ideal containing both I and J.
   (b) Prove that $I \cdot J$ is an ideal of R as well.
   (c) Suppose I is generated by $f_1, \ldots, f_r$ and J is generated by $g_1, \ldots, g_s$. Show that $I + J$ is generated by $f_1, \ldots, f_r, g_1, \ldots, g_s$.
   (d) With notation as in the previuos part, show that $I \cdot J$ is generated by all products of the form $f_i g_j$ for $i = 1, \ldots, r$ and $j = 1, \ldots, s$.
   (e) Show that the intersection $I \cap J$ is also an ideal.
2. (2 points) In the ring $\mathbb{C}[x, y]$, let I be the ideal $(xy, x + y - 1)$, and let J be the intersection of the ideals $(x - 1, y)$ and $(x, y - 1)$. First prove algebraically that $I = J$, and then prove it geometrically by comparing $V(I)$ and $V(J)$ (where you are allowed to assume I and J are radical ideals.)
3. (1 point) Find all prime ideals in $\mathbb{C}[x]/(x^2)$. Conclude that while $\mathbb{C}[x]/(x^2)$ has the same number of maximal ideals as $\mathbb{C}[x]/(x)$, it does not have the same number of prime ideals. This is reflected in the fact that if the definition of the varieties $V(x)$ and $V(x^2)$ are defined using Spec rather than $\mathrm{Spec} - \mathrm{m}$, the two varieties are different. The variety $V(x^2)$ is sometimes called a "fuzzy point" or "fat point" whereas $V(x)$ is called a "reduced point".
4. (2 points) Compute the radical of the ideal $(x^4 - 2x^2 + 1, x^4 + 2x^3 + x^2)$ in $\mathbb{C}[x]$. (Hint: First express this ideal as a principal ideal.)

Recommended practice exercises

(DO NOT hand these in - these are just extra problems I recommend you look at if you'd like more practice.)

Chapter 11 exercises 9.1, 9.3, 9.7, 9.8.

Bonus

(+1 point): Considering the quotient ring $\mathbb{C}[x, y]/(x + y, x^2 + y^2)$ as a vector space over $\mathbb{C}$, compute its dimension. Do the same for $\mathbb{C}[x, y, z]/(x + y + z, x^2 + y^2 + z^2, x^3 + y^3 + z^3)$.