Math 566: Abstract Algebra I
Homework 10

10 points total. Due Friday, Nov 5 by 11:10 am in class.

Problems

1. (2 points) Prove that the symmetric group S_n is generated by the *adjacent* transpositions $s_1, s_2, \ldots, s_{n-1}$ where $s_i = (i\ i+1)$. Show that the adjacent transpositions s_i, s_{i+1} satisfy a "braid relation":

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

   Show also that the non-adjacent transpositions, say s_i and s_j with $j \geq i+2$, commute: $s_i s_j = s_j s_i$.

2. (2 points) It is known that the relations in the previous problem are sufficient to define the symmetric group, that is:

$$S_n = \langle s_1, s_2, \ldots, s_{n-1} | s_i^2, (s_i s_{i+1})^3, (s_i s_j)^2 \rangle$$

   where the relations range over all i, j with $|i - j| > 1$. A *reduced word* for an element $w \in S_n$ is a word of *minimal length* in the generators s_i that evaluates to w. Find a reduced word for the permutation that, in list notation, is $n, n-1, n-2, \ldots, 1$, and show it is reduced. This is called the *longest word* as this element has the longest length of a reduced word. What is the length?

3. (1 point) Prove that S_n is generated by the transposition (12) along with the cycle $(123 \cdots n)$.

4. (2 points) Use the Todd-Coxeter algorithm to show that the groups $\langle x, y | x^2, y^3, xyxy \rangle$ and $\langle x, y | x^2, y^2, xyxyxy \rangle$ are both isomorphic to S_3.

5. (1 point each) Chapter 7 exercises 9.1, 10.3.

Recommended practice exercises

(DO NOT hand these in - these are just extra problems I recommend you look at if you'd like more practice.)

Chapter 7 exercise 11.3.

Bonus

(+1 point): You can label every integer either "even" or "odd" in such a way that adding two even numbers or two odd numbers gives an even number, and adding an even plus an odd gives an odd number. Can you do the same for the real numbers? In other words, can you label every real number either "E" or "O" such that the sum of any two numbers with the same label is an E number, and any E plus O is O?

If it is possible, describe all ways of doing so, and explain what this means about the set of group homomorphisms from $(\mathbb{R}, +)$ to $(\mathbb{Z}/2\mathbb{Z}, +)$.