Math 566: Abstract Algebra I
Final Exam Review

At least three of the problems on the final exam will be taken nearly-verbatim from the set of practice problems below. The other problems should give a good idea of the material that will appear on the exam. You should also review the very last homework (Homework 13), as one of the problems on the exam will be similar to a problem from Homework 13.

Problems

1. Write down the definition of a group. What additional axiom makes the group abelian?
2. Write down the definition of a ring. What additional axiom makes the ring a field?
3. Write down the definition of a group homomorphism and the definition of a ring homomorphism. What is an isomorphism in each case? What is an automorphism?
4. Define a subgroup of a group and an ideal of a ring. Give an example of an ideal of a ring that is not itself a ring.
5. State the First Isomorphism Theorem for both groups and rings.
6. Write down the class equation of S_5.
7. How many conjugacy classes are there in S_7? In A_7?
8. Let π be the permutation $8, 3, 1, 5, 6, 9, 2, 7, 4$ written in list notation.
    - Write π in cycle notation.
    - Write π^{-1} in both cycle and list notation.
    - Multiply the transposition (34) by π on the left (so, compute the product $(34)\pi$). What is the result in list notation? In cycle notation?
    - Multiply the transposition (34) by π on the left (so, compute the product $(34)\pi$). What is the result in list notation? In cycle notation?
    - Conjugate π by the transposition (34). What is the result in list notation? In cycle notation?
    - Write down the permutation matrix corresponding to π. Write down the permutation matrix of π^{-1} and verify that the two matrices are inverse matrices.
9. Use the orbit-stabilizer theorem to find the order of the symmetry group of a regular hexagonal prism (formed by drawing a regular hexagon in the $x - y$ plane, drawing another copy of the hexagon translated up to the $z = 1$ plane, and connecting the two hexagons' corresponding vertices with edges). Note that both rotational and reflective symmetries should be included.
10. Use the orbit-stabilizer theorem to find the order of the symmetry group of a regular dodecahedron.
11. Use the Sylow theorems to show that there are no simple groups of order 126.
12. Let G be a group. Prove that conjugation by an element $g \in G$ gives a group automorphism on G.

13. Let $\mathbb{Z}/n\mathbb{Z}$ be the ring of integers modulo n under $+$ and $\cdot$ taken mod n. We also consider it as its underlying abelian group with $+$ only.

    (a) Prove that there is a ring homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ if and only if $m|n$.

    (b) Prove that there is always a group homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to $\mathbb{Z}/m\mathbb{Z}$ for every n and m.

14. Show that $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/(mn\mathbb{Z})$ as groups if and only if m, n are relatively prime.

15. What is the kernel of the map $\mathbb{C}[x, y] \to \mathbb{C}[t]$ given by $x \mapsto t^2$ and $y \mapsto t^3$? Let I be this kernel. Explain how ideals of $\mathbb{C}[t]$ correspond to ideals containing I in $\mathbb{C}[x, y]$. Draw the variety $V(I)$ in two dimensions (where we restrict to the real numbers to make the drawing). How does our ring map give a parameterization of the drawing?

16. Show that the quotient ring $\mathbb{C}[x, y, z]/(x^2, y^2, z^2)$ is finite-dimensional as a complex vector space, and compute its dimension.

17. Show that the quotient ring $\mathbb{F}_2[x]/(x^3)$ is a finite set, and write out all of its elements. Which are irreducible? Also, what is its dimension as an $\mathbb{F}_2$-vector space? How could this help you compute the number of elements?

18. Show that $\mathbb{R}[x]/(x^3 + 3)$ is not isomorphic to $\mathbb{R}[x]/(x^3)$ as a ring. (Hint: it is NOT enough to show that two polynomials multiply differently; one needs to show that some fundamental property of the rings is different and therefore there cannot exist any isomorphism, even a non-obvious one.)