Counting with Group actions

Def: An *action* of a group G on a set X is a map

$$G \times X \to X$$
$$(g, x) \mapsto g \cdot x$$

notation

such that $h \cdot (g \cdot x) = (hg) \cdot x$

multiplication in G.

Ex: Representations are actions on a vector space ($G \times V \to V$), where each map $M_g : V \to V$ is linear.

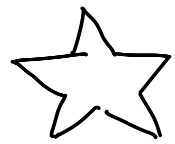
Ex: S_n acts on $[n] = \{1, 2, 3, \ldots, n\}$ by

$$\pi \cdot i = \pi(i)$$

Ex: S_n acts on itself by

- Conjugation: $\pi \cdot \sigma = \pi \sigma \pi^{-1}$
- Left multiplication: $\pi \cdot \sigma = \pi \sigma$

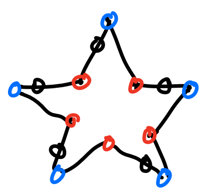
Ex: $S_3 \times S_2$ acts on $\{1, 2, 3, 4, 5\}$ by S_3 acting on $\{1, 2, 3\}$ and S_2 acting on $\{4, 5\}$

Ex: $\mathbb{Z}/5\mathbb{Z}$ acts on $\longrightarrow$ 
("cyclic perms in S_5")
by rotation.

Def: The *orbit* of $x \in X$ under an action $G \curvearrowright X$ is $\{g \cdot x \mid g \in G\} =: \mathrm{Orb}_G(x)$

Note: $\mathrm{Orb}_G(x) = \mathrm{Orb}_G(gx)$ for any g.
Being in the same orbit is an equivalence relation, so the orbits partition the set.

Ex: Orbits of $\mathbb{Z}/5\mathbb{Z}$ acting on star look like:



(Red, blue, and black are all orbits)

Ex: Only one orbit in $S_n \curvearrowright [n]$

Two orbits in $S_3 \times S_2 \curvearrowright [5]$

One orbit in $S_n \curvearrowright S_n$ by left mult.

$p(n)$ orbits in $S_n \curvearrowright S_n$ by conjugation
(conjugacy classes)

Def: The *stabilizer* of $x \in X$ under $G \curvearrowright X$ is $\{g \in G : gx = x\} =: \mathrm{Stab}_G(x)$

Ex: $\text{Stab}_{\mathbb{Z}/5\mathbb{Z}}(\text{top pt of star}) = \{id\}$

$\text{Stab}_{S_n}(n) = S_{n-1}$

$\text{Stab}_{S_5}((123)(45)) = S_3 \times S_2$

conjugation action

$\text{Stab}_{S_3 \times S_2}(1) = S_2 \times S_2$

permute 2,3

permute 4,5

Lemma: $\text{Stab}_G(x)$ is a subgroup of G. (homework)

Thm: (Orbit-Stabilizer): For $x \in X$,

$$|\text{Stab}_G(x)| \cdot |\text{Orb}_G(x)| = |G|.$$

Pf: Let $s = |\text{Stab}_G(x)|$.

Now, consider all the elements of G that send x to $gx = y$. If $hx = gx$ then $g^{-1}hx = x$ so $g^{-1}h \in \text{Stab}(x)$, $\Rightarrow$ $h = g \cdot r$ for some $r \in \text{Stab}(x)$. So there are s elements of G sending x to y for every $y \in \text{Orb}_G(x)$.

Thus $s \cdot |\text{Orb}_G(x)| = G$ as desired. $\square$

Ex: $|\text{Orb}_{S_n}(\underbrace{(12\cdots\lambda_1)(\lambda_1+1\cdots\lambda_1+\lambda_2)\cdots}_{=\pi_\lambda})| = |\text{conj class of cycle type } \lambda|$

(↑ conjugation)

$$= \frac{|S_n|}{|\text{Stab}_{S_n}(\pi_\lambda)|}$$

$$= \frac{n!}{\prod i^{m_i}\, m_i!}$$

choose which elt each i-cycle starts with

reorder the m_i cycles of size i

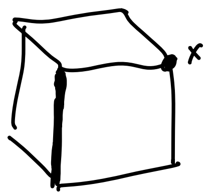
$$= \frac{n!}{z_\lambda}.$$

Ex: $S_n \curvearrowright \{1,2,\ldots,n\}$:

$$\left.\begin{array}{l} |\text{Stab}_{S_n}(n)| = (n-1)! \\ |\text{Orb}_{S_n}(n)| = n \end{array}\right\} \text{ product is } n!$$

Ex: Can compute sizes of groups using this as well. How many rotations fix a cube?

Such rotations form the <u>rotation group</u> G of a cube under composition.



G acts on the vertices of a cube; how many stabilize a fixed vertex x?

$|Stab(x)| = 3$

$|Orb(x)| = \#\text{ vertices} = 8$ (roll a die)

$$\Rightarrow |G| = 3 \cdot 8 = 24$$

Add reflections $\rightsquigarrow$ 48

<u>Ex:</u> We'll show $\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$ using Orbit-Stabilizer:

Let S_n act on the set of all k-elt subsets of $[n]$. For instance,

$$(124)(67)\cdot\{1,4,5,6\} = \{2,1,5,7\}$$

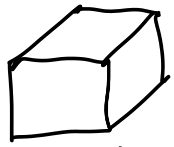
Then $\left|Orb_{S_n}(\{1,2,\ldots,k\})\right| = \binom{n}{k}$

$$\left|Stab_{S_n}(\{1,2,\ldots,k\})\right| = |S_k|\cdot|S_{n-k}| = k!\,(n-k)!$$

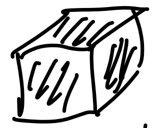
So $\binom{n}{k}\cdot k!\cdot(n-k)! = n!$. QED

Burnside's Lemma (Orbit counting)

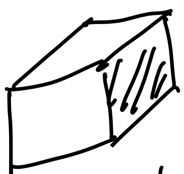
Q: How many ways can we two-color the faces of a cube up to rotation?



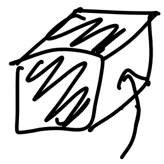
all white



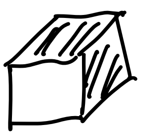
all black



One black face

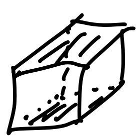


one white face



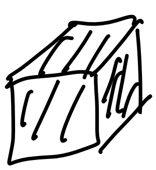
two adjacent black

→ switch colors

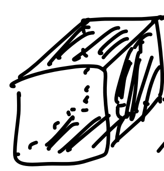


two nonadjacent black

→ switch colors



three adjacent



three around

There are <u>10</u>. Short cut:

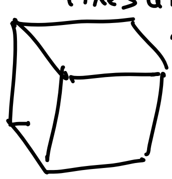
<u>Lemma</u>: The number of orbits of $G \curvearrowright X$ is

$$\frac{1}{|G|}\sum_{\pi\in G} \underbrace{\#\{x\in X : \pi x = x\}}_{= |\mathrm{Fix}(\pi)|}$$

<u>Ex</u>: $$\frac{1}{24}\Big(1\cdot 2^6 + 8\cdot 2^2 + 3\cdot 2\cdot 2^3 + 6\cdot 2^3$$

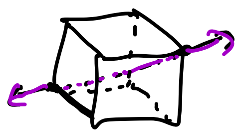
identity

fixes all 2^6 colorings



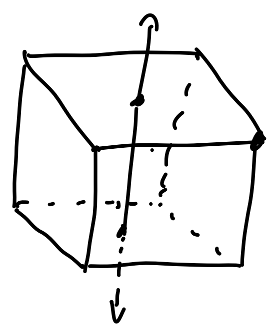
two non-id rotations through each pair of opposite vertices

Each fixes 2^2 colorings



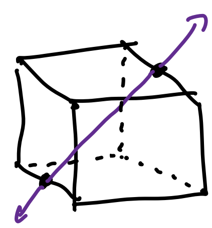
6 90° rotations through face midpoint axes

each fixes 2^3 colorings



6 non-id rotations through edge midpt axes

each fixes 2^3 colorings



$$+ 3\cdot 2^4\Big)$$

3 180° rotations through faces

$$= \frac{1}{24}\left(64 + 32 + \quad 18 \cdot 8\right) = \frac{1}{24}\left(32 \cdot 3 + 6 \cdot 24\right)$$

$$= 4 + 6 = 10$$

<u>Proof:</u> Consider the set of all pairs

$$\{(g, x) : g \in G,\ x \in X,\ gx = x\}.$$

We'll count in two ways.

First, for each g, there are $|\mathrm{Fix}(g)|$ pairs starting with g, so the set has size

$$\sum_{g \in G} |\mathrm{Fix}(g)|.$$

On the other hand, for each x, there are $|\mathrm{Stab}_G(x)|$ pairs ending in x, so the set has size

$$\sum_{x \in X} |\mathrm{Stab}_G(x)| = \sum_{x \in X} \frac{|G|}{|\mathrm{Orb}_G(x)|}$$

$$= |G| \sum_{x \in X} \frac{1}{|\mathrm{Orb}_G(x)|}$$

$$= |G| \sum_{\substack{\mathcal{O} \text{ orbit} \\ \text{of } G \curvearrowright X}} \sum_{x \in \mathcal{O}} \frac{1}{|\mathcal{O}|}$$

$$= |G| \sum_{\substack{\mathcal{O} \text{ orbit} \\ \text{of } G \curvearrowright X}} \frac{|\mathcal{O}|}{|\mathcal{O}|}$$

$$= |G| \sum_{\mathcal{O} \text{ orbit}} 1$$

$$= |G| \cdot \# \text{ orbits.}$$

QED

Ex 2: $S_n \curvearrowright \{\text{binary sequences length } n\}$

$\#$ orbits $= n+1$ (based on number of 1's)

Each permutation π, if it has cycle type λ, fixes $2^{\ell(\lambda)}$ elements.

(123)(45)(67) fixes $\underline{000}\,\underline{00}\,\underline{00}$, $\underline{000}\,\underline{00}\,\underline{11}$, etc

Thus

$$n+1 = \frac{1}{n!} \sum_{\lambda \vdash n} \left(\frac{n!}{z_\lambda}\right) 2^{\ell(\lambda)}$$

$$\Rightarrow \boxed{n+1 = \sum_{\lambda \vdash n} \frac{2^{\ell(\lambda)}}{z_\lambda}}$$

Interesting identities!

Comes from:

$$s_{(4)}(x_1,x_2) = \sum \frac{p_\lambda(x_1,x_2)}{z_\lambda}$$

Plug in $x_1 = x_2 = 1$.

Ex: $n=4$

$$5 = \frac{2^{\ell(4)}}{z_{(4)}} + \frac{2^{\ell(3,1)}}{z_{(3,1)}} + \cdots$$

$$= \frac{2}{4} + \frac{2^2}{3} + \frac{2^2}{2^2 \cdot 2!} + \frac{2^3}{2 \cdot 2!} + \frac{2^4}{4!}$$

$$= \frac{1}{2} + \frac{4}{3} + \frac{1}{2} + 2 + \frac{2}{3}$$

$$= 1 + 2 + 2 = 5$$

Ex 3: How many distinct necklaces can you make (up to rotation) with red, green, blue beads?

- Using 7 total beads? $\frac{1}{7}(3^7 + 6\cdot 3) = 315$
- Using 6 total beads?

$$\frac{1}{6}\left(3^6 + 2\cdot 3^2 + 2\cdot 3 + 3^3\right)$$

$$= \frac{1}{2}\left(3^5 + 2\cdot 4 + 9\right)$$

$$= \frac{1}{2}\left(24 + 252\right) = 4 + 126 = 130$$

The Cyclic Sieving Phenomenon

- Inclusion/exclusion and sign reversing involutions dealt w/ sums of ± 1.
- What about sums involving complex nth roots of unity?

Def: $\omega_n = e^{2\pi i/n}$ nth root of unity

Note: $\omega_n^n = 1 \Rightarrow 1 - \omega^n = 0 \Rightarrow \underbrace{(1-\omega)(1+\omega+\omega^2+\dots+\omega^{n-1})}_{=0}$

$$\Rightarrow \boxed{1 + \omega + \omega^2 + \dots + \omega^{n-1} = 0}$$

no partial sum here is 0 if ω_n a primitive nth root of unity.

Def. A triple (X, G, f) exhibits the *cyclic sieving phenomenon* if

- X is a set, G is a group, $G \curvearrowright X$
- f is a polynomial $f(q) \in \mathbb{N}(q)$ s.t. for all $g \in G$,

$$|\mathrm{Fix}_X(g)| = f(\omega_{o(g)})$$

where $o(g)$ is the *order* of g in G: smallest $k > 0$ s.t. $g^k = 1$.

Ex: $G = \mathbb{Z}/n\mathbb{Z} = \langle (1\,2\,3\,\cdots\,n) \rangle \subseteq S_n$

acting on $X = \left(\!\binom{[n]}{k}\!\right) = \{$multisets of size k from $[n]\}$

e.g. $X = \{11, 12, 13, 22, 23, 33\} = \left(\!\binom{[3]}{2}\!\right)$

Claim: $f(q) = \left(\!\binom{n}{k}\!\right)_q = \binom{n+k-1}{k}_q$ exhibits the cyclic sieving phenomenon for X, G.

In $n=3$, $k=2$ setting:

$$\left(\!\binom{3}{2}\!\right)_q = \binom{4}{2}_q = \frac{(4)_q!}{(2)_q!(2)_q!} = \frac{(4)_q \cdot (3)_q}{(2)_q} = (1+q^2)(1+q+q^2)$$

$$= 1 + q + 2q^2 + q^3 + q^4$$

(123) has order 3,

$|\mathrm{Fix}((123))| = 0 \quad \rightsquigarrow \quad 1 + \omega_3 + 2\omega_3^2 + \omega_3^3 + \omega_3^4 = 0$ ✓

$|\mathrm{Fix}(id)| = 6$ $\rightsquigarrow$ plug in 1 ✓

<u>For $n=4$, $k=2$</u>: $\left(\!\binom{4}{2}\!\right)_q = \binom{5}{2}_q = \frac{(5)_q(4)_q}{(2)_q}$

$= (1+q+q^2+q^3+q^4)(1+q^2)$

(1234) has order 4, plugging in 4th root $= 0$

no fixed points ✓

But $(1234)^2 = (13)(24)$ has order 2, fixed points:

$13, \; 24$ $\leftarrow$ two fixed pts ✓

Plug in a primitive "square root of unity" $= -1$,

$(1-1+1-1+1)(1+1) = 1 \cdot 2 = 2$ ✓

<u>Lemma</u>: Let $g \in G = \langle (12\dots n) \rangle$ have order d.

then $|\mathrm{Fix}_{\left(\!\binom{[n]}{k}\!\right)}(g)| = \begin{cases} \left(\!\binom{n/d}{k/d}\!\right) & \text{if } d \mid k \\ 0 & \text{otherwise.} \end{cases}$

<u>Pf:</u> If $d \nmid k$, g can't map a k-elt multiset to itself, since it cycles elts. So 0 in this case.

If $d | k$, the fixed points are disjoint unions of cycles of g (repeated cycles allowed) and each cycle has length d, so there are n/d to choose from and we choose k/d of them to make a multiset of size k. $\square$

Then, computations show: (Sagan)

① $$\lim_{q \to \omega_d} \frac{[m]_q}{[n]_q} = \begin{cases} \frac{m}{n} & d | n \\ 1 & \text{else} \end{cases}$$

② $$f(\omega_d) = \begin{cases} \left(\!\binom{n/d}{k/d}\!\right) & d | k \\ 0 & \text{else.} \end{cases}$$