Designs

Problem in agriculture: (Experimental designs)

- $\underline{v}$ different varieties of grain, want to compare them

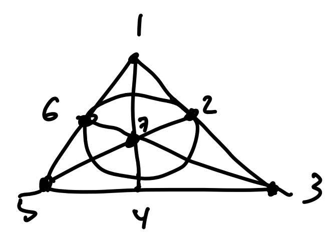
  Grain types: 1, 2, 3, 4, 5, 6, 7

- Also want to vary conditions - soil, temperature, etc, so you make $\underline{b}$ blocks, each of the same size k, in terms of number of grains tested, in different conditions

  $k=3$: $\underset{\text{soil A}}{\{1,2,4\}}$, $\underset{\text{Soil B}}{\{2,3,5\}}$, ... $b=7$ (7 types of soil)

- Too inefficient to test every grain in every soil, but want to compare all $\underset{\uparrow\, \underline{t}=2}{\underline{\text{pairs}}}$ of grains to determine the best one:

  Soil A: 123
  Soil B: 345
  561
  ⋮
  147
  257
  367
  246



"Fano plane"

7 blocks, each pair of grains appears together in exactly $\underline{1}$ block. Each grain is tested $\underline{3}$ times

- In general have each pair occur in $\underline{\lambda}$ blocks each. Each variety occurs in $\underline{r}$ blocks each.

<u>Def</u>: A <u>t-(v,k,λ) design</u> is a collection of blocks $\mathcal{B}$ consisting of k-elt subsets of a set X of size v (elts of X are called <u>points</u>) s.t.

* Every t-elt subset of X is in exactly λ blocks.

<u>Ex</u>: The Fano plane above is a 2-$(7,3,1)$ design.

<u>Def</u>: For $\lambda=1$, called a <u>Steiner system</u>.

Note: We'll be assuming repeated blocks are not allowed.

<u>Ex</u>: $\binom{[n]}{k}$ is a t-(n,k,λ) design for what λ? <u>Ans</u>: $\lambda=\binom{n-t}{k-t}$.

<u>Lemma</u>: Any t-design is also an s-design for any $s\le t$.

<u>Pf</u>: Let $(X,\mathcal{B})$ be a t-(v,k,λ) design, and let $s\le t$. Then each s-size subset $S\subseteq X$ is contained in exactly $\binom{v-s}{t-s}$ sets $T\subseteq X$ of size t. Each T occurs in λ blocks $B\in\mathcal{B}$, so this gives $\lambda\binom{v-s}{t-s}$ occurrences of S in blocks B, where we have overcounted by the number of T containing S within any given block B containing S, which is $\binom{k-s}{t-s}$.

So $(X,\mathcal{B})$ is an s-(v,k,λ') design where $\lambda'=\lambda\dfrac{\binom{v-s}{t-s}}{\binom{k-s}{t-s}}$.

Lemma: Let r be the number of blocks each $x \in X$ occurs in, in a design. Then

$$bk = vr$$

"Burger King = Virtual Reality"

Pf: $b \cdot k = \#$ pairs (x, B) : $x \in B$, $B \in \mathcal{B}$

$= v \cdot r$

(note: r exists by previous lemma)

Lemma: In a 2-design, have $r(k-1) = (v-1)\lambda$.

Pf: $r = \#$ blocks $x \in X$ is contained in

$k-1 = \#$ ways to pick another elt y to pair w/ x in a block B

$v-1 = \#$ elts y besides x in X

$\lambda = \#$ blocks containing (x, y)

Fix $x \in X$. $(v-1)\lambda = \#$ pairs (y, B) s.t. $x, y \in B$.

$= r(k-1)$.

Def: Incidence matrix of a design: Write each block as a 0-1 row vector.

Ex: Fano plane is

$$\begin{array}{c}
\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\
\begin{pmatrix}
1 & 1 & 1 & & & & \\
 & & 1 & 1 & 1 & & \\
1 & & & & 1 & 1 & \\
1 & & & 1 & & & 1 \\
 & 1 & & & 1 & & 1 \\
 & & 1 & & & 1 & 1 \\
 & 1 & & 1 & & 1 &
\end{pmatrix}
\end{array}$$

Can rearrange rows:

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

symmetric matrix!

<u>Def</u>: The <u>dual</u> of a design $\mathcal{D} = (X, \mathcal{B})$ is written $\mathcal{D}^T$ and consists of transposing the incidence matrix, i.e.

$$X^T = \mathcal{B}$$
$$\mathcal{B}^T = \{\{B \ni x\} : x \in X\}$$

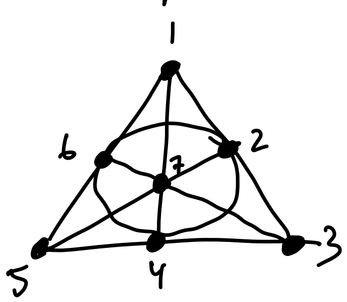
<u>Symmetric</u> design if $\mathcal{D}^T \cong \mathcal{D}$.

isomorphic

<u>Isomorphism</u>: $(X, \mathcal{B}) \longrightarrow (X', \mathcal{B}')$ is a bijection $X \to X'$ that induces a bijection $\mathcal{B} \to \mathcal{B}'$.

<u>Note</u>: There are nontrivial <u>automorphisms</u> - isomorphisms to itself.

<u>Ex</u>: Automorphism of the Fano plane:



$6 \leftrightarrow 2$
$5 \leftrightarrow 3$.

Ex: A 2-$(9,3,1)$ design:

$$\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

123
456
789
147
258
369
168
249
357
159
267
348

Dual is not a 2-design, but is a 1-design;

$\boxed{\text{The ual of a 1-design is a 1-design}}$

Above ex is a 1-$(9,3,4)$ design

Its dual is a 1-$(12,4,3)$ design.

In fact, 2-designs' duals are 2-designs iff their incidence matrix is square, i.e. $b=v$:

Thm (Fisher's Inequality) In a 2-design with $k<v$, we have $b \geq v$.

Pf: First note that if M is the inc. matrix,

$$M = \begin{array}{c} \\ B_1 \\ \vdots \\ B_b \end{array} \begin{array}{c} x_1 \quad \cdots \quad x_v \\ \begin{pmatrix} & & 1 & & \\ & 1 & & & \\ & & & \ddots & \\ & & & & \end{pmatrix} \end{array}$$

$M^T M$ = matrix whose (i,j) entry is the dot product of column i w/ col j in M

= matrix whose (i,j) entry is

$$\begin{cases} \#\ B \text{ containing } x_i \text{ and } x_j & \text{if } i \neq j \\ \#\ B \text{ containing } x_i & \text{if } i = j \end{cases}$$

$$= \begin{cases} \lambda & \text{if } i \neq j \\ r & \text{if } i = j \end{cases}$$

$$= (r-\lambda) I_v + \lambda J_v .$$

$$\boxed{J_v = \overbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}}^{v}}$$

Want to show $M^T M$ is nonsingular, which would show $b \geq v$. So want to compute

$$\det(M^T M) = \det\big((r-\lambda) I_v + \lambda J_v\big)$$

$$= \text{product of eigenvalues.}$$

The all-1's vector $u = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector

w/ eigenvalue $r-\lambda+\lambda v = r+\lambda(v-1) = r+r(k-1)$
$= rk$.

The orthogonal vectors to u, like $\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$,

are also eigens w/ eigenvalue $r-\lambda$

$\Rightarrow \det = rk(r-\lambda)^{v-1}$.

To check $r-\lambda \neq 0$: we have

$$r(k-1) = (v-1)\lambda$$

$$(r-\lambda)(k-1) = \lambda(v-1) - \lambda(k-1)$$

$$(r-\lambda)(\underbrace{k-1}_{>0}) = \lambda(\underbrace{v-k}_{>0}) \quad \text{by assumption}$$

so this is >0.

Thus $\det M^T M \neq 0$, so $b \geq v$.

<u>Thm</u>: TFAE (and such a 2-design is called <u>square</u>) for a 2-design w/ $k<v$:

a) $b=v$
b) $r=k$
c) Any two blocks have λ common points
d) Any two blocks have a constant number of common points

Pf: (a) $\Leftrightarrow$ (b) by $bk = vr$

(b) $\Rightarrow$ (c): If $r = k$, then $MJ_v = J_r M = kJ_{r\times v}$

So
$$
\begin{aligned}
MM^T &= M(M^T M)M^{-1} \\
&= M((r-\lambda)I_v + \lambda J_v)M^{-1} \\
&= ((r-\lambda)I_r + \lambda J_r)MM^{-1} \\
&= (r-\lambda)I_r + \lambda J_r
\end{aligned}
$$

and so any two blocks have λ pts in common.

(c) $\Rightarrow$ (d) trivially

(d) $\Rightarrow$ (a): If (d) is true then the dual design is a 2-design, and so $b \geq v$ and $v \geq b$ and so $b = v$.

Hadamard designs

Def: A Hadamard matrix is an $n \times n$ matrix with ± 1 entries s.t. $HH^T = I_n$.

i.e. each row $\vec{r}$ has $\langle \vec{r}, \vec{r} \rangle = n$ and two rows $\vec{r}_1 \neq \vec{r}_2$ have $\langle \vec{r}_1, \vec{r}_2 \rangle = 0$.

Ex:
$$
\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\qquad
\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}
\qquad
\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}
$$

(1)

3×3 Hadamard? $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$ X

$\rightsquigarrow$ must have even size (after 1)

Why Hadamard matrices? Equality case in "Hadamard det thm": if all entries of M have $|M_{ij}| \leq 1$,

$$|\det(M)| \leq n^{n/2}.$$

Note: Can multiply any row or col by -1 and still have a Hadamard matrix. Also swap rows or columns. Normalized: $\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & & & & \\ \vdots & & ?? & & \\ 1 & & & & \end{pmatrix}$

Ex: Normalization of $\begin{pmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{pmatrix}$ is

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

__Lemma:__ Hadamard matrices must have $n = 1, 2, 4t$.

__Pf.__ Suppose H Hadamard, size > 3, consider first 3 rows, normalize and arrange cols so they look like:

$$\left(\begin{array}{c|c|c|c}
\underbrace{\begin{matrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ & & & & \\ & & & & \end{matrix}}_{x} &
\underbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ & & & \\ & & & \end{matrix}}_{y} &
\underbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \\ & & & \\ & & & \end{matrix}}_{z} &
\underbrace{\begin{matrix} 1 & 1 & \cdots & 1 \\ -1 & -1 & \cdots & -1 \\ -1 & -1 & \cdots & -1 \\ & & & \\ & & & \end{matrix}}_{w}
\end{array}\right)$$

Then $x + y = z + w$ and $x + z = y + w$ and

$x + w = y + z$

So $x = y = z = w$.

Thus n is div. by 4.

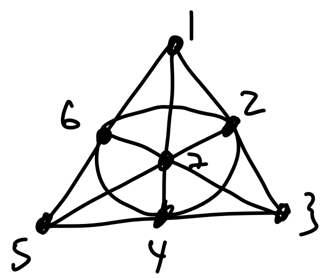
__Conjecture__ (OPEN!) There $\exists$ a Hadamard matrix of order $4t$ for every t.

Connection to designs

Prop: Let H be a normalized Hadamard matrix (first row and first column all 1's). Let M be the matrix formed by deleting the first row, col from H and replacing every $+1$ by 1 and -1 by 0.
Then M is the incidence matrix of a 2-$(4t-1, 2t-1, t-1)$ design.
If we instead replace $+1$ by 0 and -1 by 1, get a 2-$(4t-1, 2t, t)$ design.

Ex:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \end{pmatrix}$$



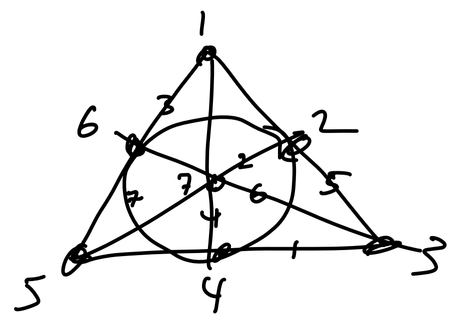
Hadamard matrix order 8
$(t=2)$

<u>Duality</u> - an isom. between $\mathcal{D}$ and its dual (for a symmetric design $\mathcal{D}$ with $\mathcal{D} \cong \mathcal{D}^T$): a pair of bijections

$\sigma : X \to \mathcal{B}$

$\tau : \mathcal{B} \to X$

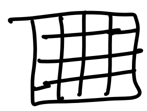
s.t. $x \in B$ iff $\tau(B) \in \sigma(x)$

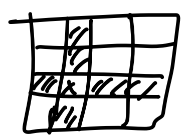


polarity

If $\sigma\tau = id_{\mathcal{B}}$ and $\tau\sigma = id_X$, say the duality is a ~~polarity~~.

<u>Ex:</u> A 2-(16,6,2)-design:

$X = \{$ unit squares in:  $\}$

$\mathcal{B} = \{$ blocks of the form  $\}$

(6 shaded squares in vertical or horz alignment w/ a given square x.)

<u>Polarity</u>: $x \to$ its associated block.

In this polarity, no x is in its own block.

Strongly Regular Graphs (SRG's)

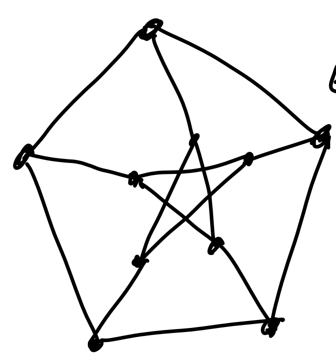
Regular graph – all vertices the same degree

Def: An SRG w/ parameters (n, k, λ, μ) is a graph w/ n vertices, not K_n or its complement, s.t. # common neighbors of any two vertices x, y is

$$\begin{cases} k & \text{if } x = y \\ \lambda & \text{if } x \text{ adj to } y \\ \mu & \text{if } x \text{ not adj. to } y \end{cases}$$

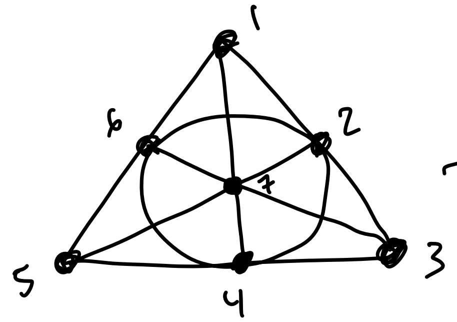
← regular deg k

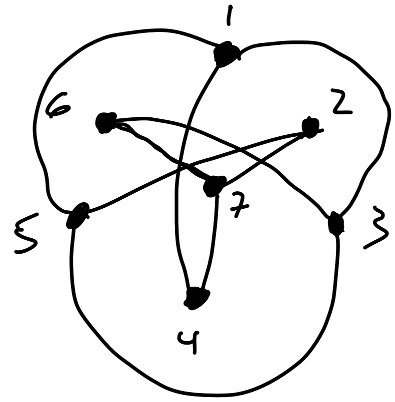
Ex:



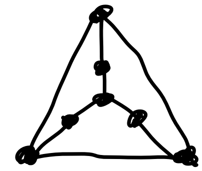
← Petersen Graph; SRG w/ parameters $(10, 3, 0, 1)$

Construction: Given a design w/ a polarity, make a graph on X where we join $x - y$ if $x \neq y$ and $y \in \sigma(x)$





$\cong$



6, 4, 2 degree 2
1, 5, 7, 3 degree 3

Not regular in this example because sometimes $x \in \sigma(x)$ and other times not. But:

Lemma: If, in a polarity σ, we never (or always) have $x \in \sigma(x)$, the associated graph is strongly regular.

Pf: For never: We have $|\sigma(x) \cap \sigma(y)| = \lambda$ for all $x \neq y$ (where the design is a symmetric $2-(v,k,\lambda)$ design). So whether or not x, y are adjacent, they have λ common neighbors. It also has degree k, so it is an (n, k, λ, λ) SRG.

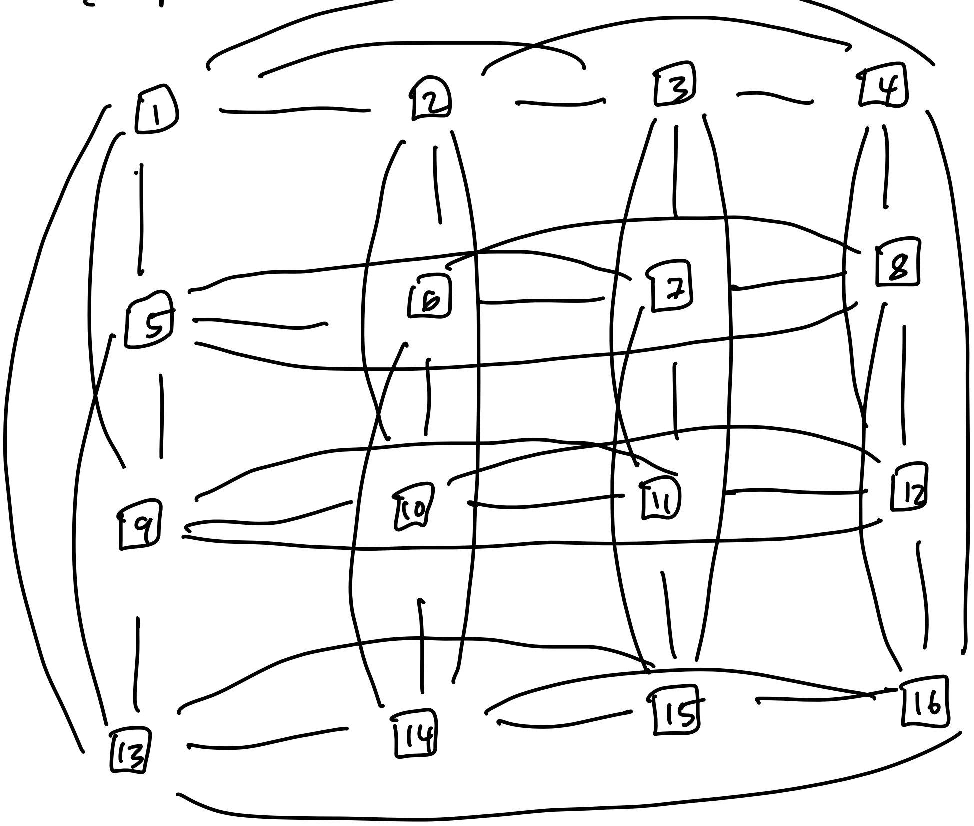
Ex:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

$X = \{$squares at left$\}$

$\mathcal{B} = \{2\,3\,4\,5\,9\,13,\ 1\,3\,4\,6\,10\,14,\ \dots\}$

Graph:



$L_2(4)$

<u>Converse</u>: Given an SRG $^{(V,E)}$ of parameters (n, k, λ, λ), there is a symmetric $2-(n, k, \lambda)$ design w/ a polarity that gives rise to it.

<u>Pf:</u> Let $X = V$ and define the blocks to be the sets of neighbors of each $x \in X$.

This gives the $2-(n, k, \lambda)$ design, and the polarity sends each x to its neighbor block. $\square$