Characters and the Murnaghan–Nakayama rule:

Def: The *character* χ_V of a representation V of G w/ matrices $g \mapsto M_g$ acting on V is

$$\chi_V : G \to \mathbb{C}$$
$$\chi_V(g) = \mathrm{tr}(M_g).$$

$\uparrow$ trace = sum of diagonal entries

(Note: basis-independent)

Lemma: χ_V is constant on conjugacy classes:

Pf:
$$\chi_V(hgh^{-1}) = \mathrm{tr}(M_{hgh^{-1}}) = \mathrm{tr}(M_h M_g M_h^{-1})$$
$$= \mathrm{tr}(M_g) \quad \text{sum of eigenvalues}$$
$$= \chi_V(g)$$

Thus we can make a *character table* for G:

	conj classes
χ's of irred. reps	

$\leftarrow$ square table.

Ex:

S_2	id	(12)
$\chi_{(2)}$	1	1
$\chi_{(1,1)}$	1	-1

S_3	id	$[(12)]$	$[(123)]$
$\chi_{(3)}$	1	1	1
$\chi_{(1,1,1)}$	1	-1	1
$\chi_{(2,1)}$	2	0	-1

$$(12)\begin{pmatrix}1\\-1\\0\end{pmatrix} \to \begin{pmatrix}-1\\1\\0\end{pmatrix} = -v_1$$

$$(12)\begin{pmatrix}0\\1\\-1\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix} = v_1+v_2 \qquad \begin{array}{c|cc} & v_1 & v_2 \\ \hline v_1 & \multicolumn{2}{c}{\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}} \\ v_2 & & \end{array} \rightsquigarrow tr = 0$$

Why do we like characters?

① Additive across $\oplus$: $\chi_V + \chi_W = \chi_{V\oplus W}$.

<u>Ex:</u> $\chi_{\text{perm. rep}} = \chi_{(3)} + \chi_{(2,1)} =$

id	(12)	(123)
3	1	0

$$\begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix} \begin{pmatrix}&1&\\1&&\\&&1\end{pmatrix} \begin{pmatrix}&1&\\&&1\\1&&\end{pmatrix}$$

② Can uniquely determine the rep. from its character (and decompose into irreducibles)

Alt def of Frobenius

<u>Def:</u> Let V be a rep of S_n.

$$\mathrm{Frob}(V) := \frac{1}{n!}\sum_{\pi\in S_n} \chi_V(\pi)\, p_{c(\pi)}$$

power sum (p) — cycle type ($c(\pi)$)

<u>Lemma:</u>
$$= \sum_{\lambda \in S_n} \chi_V(\lambda \text{ conj class}) \frac{p_\lambda}{z_\lambda}$$

<u>Proof:</u> Size of the conjugacy class w/ cycle type λ is $\dfrac{n!}{z_\lambda}$ where, if

$m_i = \#i\text{'s in } \lambda, \qquad z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2!\, 3^{m_3} m_3! \cdots$

<u>Ex:</u> $\mathrm{Frob}(V_{(2,1)}) = \underset{\substack{\lambda=(1,1,1)\\ \mathrm{id}}}{2 \cdot \dfrac{p_{(1,1,1)}}{3!}} + \underset{\substack{\lambda=(2,1)\\ (12)}}{0 \cdot \dfrac{p_{(2,1)}}{2}} - \underset{\substack{\lambda=(3)\\ (123)}}{\dfrac{p_{(3)}}{3}}$

$= \frac{1}{3} p_{(1,1,1)} - \frac{1}{3} p_{(3)})$

$= \frac{1}{3}\left(p_1^3 - p_3 \right)$

$= \frac{1}{3}\left((x_1 + x_2 + x_3 + \cdots)^3 - x_1^3 - x_2^3 - x_3^3 - \cdots \right)$

$= m_{(2,1)} + 2 m_{111}$

$= s_{(2,1)}$

$\mathrm{Frob}(V_{(3)}) = \dfrac{p_{(1,1,1)}}{3!} + \dfrac{p_{(2,1)}}{2} + \dfrac{p_{(3)}}{3} = s_{(3)}$

$\mathrm{Frob}(V_{(1,1,1)}) = \dfrac{p_{(1,1,1)}}{3!} - \dfrac{p_{(2,1)}}{2} + \dfrac{p_{(3)}}{3} = s_{(1,1,1)}$

<u>Thm:</u> The character tables for each S_n are the transition matrices from the $\{s_\lambda\}$ basis to the $\{p_\lambda / z_\lambda\}$ basis.

Pf in Stanley. For now we will simply combinatorially derive the Murnaghan - Nakayama rule - a combinatorial rule for computing the numbers $\chi_\mu(\lambda) := \chi_{V_\mu}(\pi \text{ of cycle type } \lambda)$.

$$= \text{coeff of } \frac{p_\lambda}{z_\lambda} \text{ in } s_\mu$$

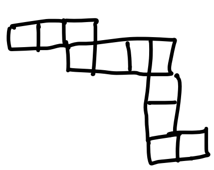
dual to p basis

$$= \langle s_\mu, p_\lambda \rangle$$

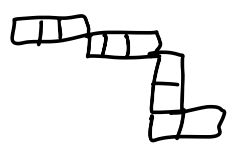
$$= \text{coeff of } s_\mu \text{ in } p_\lambda .$$

So, suffices to compute the Schur expansion of $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}$.

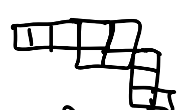
Def: A border strip or ribbon is a connected skew shape w/ no 2×2 square:



connected



not connected.



not a ribbon.

__Lemma:__ $$s_\mu p_r = \sum_{\substack{\lambda:\ \lambda/\mu \\ \text{border} \\ \text{strip size } r}} (-1)^{ht(\lambda/\mu)} s_\lambda$$

where $ht(\lambda/\mu) = (\#\text{rows of } \lambda/\mu) - 1$.

__Note:__ This is similar to the Pieri rules for adding on horz or vertical strips when mult. by h_r or e_r resp. For p_r we add a border strip.

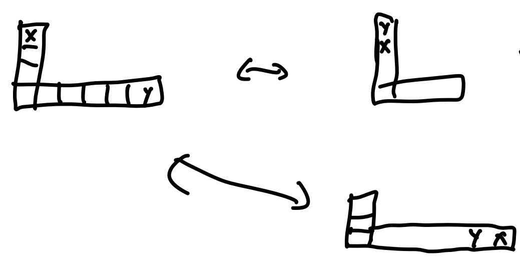
__Pf:__ Let's first show it for $\mu = \emptyset$:

$$p_r = \sum_{\substack{\text{hooks} \\ \text{size } r}} (-1)^{\#\text{squares above bot}} s_{\text{hook}} \,?$$

__Ex:__ $p_4 = s_{(4)} - s_{(3,1)} + s_{(2,1,1)} - s_{(1,1,1,1)}$

Except for $\boxed{1\,1\,1\,1}$, $\boxed{2\,2\,2\,2}$, etc,

each hook SSYT can cancel w/ one to its right or left by comparing top and right elts:



if $y > x$

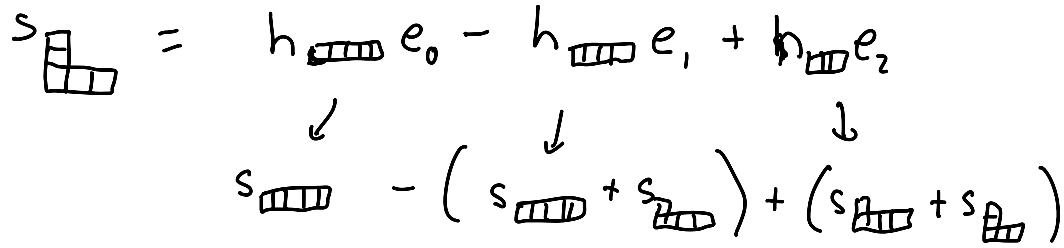
if $x \geq y$

So all terms cancel except for p_r.

Now: $p_r = \sum_{j=0}^{r-1} (-1)^j s_{(r-j,\,1^j)}$

Notice: $s_{(r-j,\,1^j)} = \sum_{k=0}^{j} (-1)^{j-k} h_{r-k} e_k$

by Pieri rules:



Thus $p_r s_\mu = \sum_{j=0}^{r-1} \sum_{k=0}^{j} (-1)^k h_{r-k} e_k s_\mu$

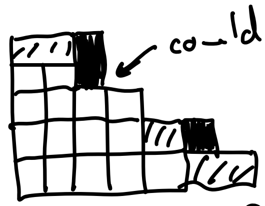
add a horizontal strip

add a vertical strip

$= \sum_{r-1 \geq j > k \geq 0} \sum_{\substack{\nu/\mu \\ \text{horz + vert} \\ r-k \quad k}} (-1)^k s_\nu$

horz strip + vertical strip can make a ribbon,

sometimes disconnected though:



could be in horz or vertical strips

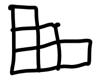
this box could have been in horz or vertical strip $\Rightarrow$ this shape is counted twice in inner sum.

(Sketch:)

Sign-reversing involution: switch bottom-right box from horz to vertical to cancel all but $j=k$ terms,
then if disconnected, switch second "bottom of connected cpt" square.

This cancels all disconnected ribbons and leaves us with the desired result.

<u>Cor</u>: $$P_\lambda = \sum_{\substack{\text{Border} \\ \text{strip tableaux} \\ \text{shape } \mu \\ \text{content } \lambda_1, \lambda_2, \ldots}} (-1)^{\sum \text{hts of border strips}} s_\mu$$

<u>Ex</u>: Coeff of s  in $P_{(3,3)}$:

$$\begin{array}{lll} 2 & & \\ 2 & 2 & \\ 1 & 1 & 1 \end{array} \quad , \quad \begin{array}{lll} 1 & & \\ 1 & 2 & \\ 1 & 2 & 2 \end{array}$$

$$\downarrow \qquad\qquad \downarrow$$

$$(-1)^{0+1} + (-1)^{2+1} = -2$$