Recall: $\Lambda_{\mathbb{Q}}(x_1, x_2, \ldots)$ = ring of symmetric functions in $x_1, x_2, \ldots$ w/ coefficients in $\mathbb{Q}$

Write $\Lambda = \Lambda_{\mathbb{Q}}(x_1, \ldots)$ for class today

Note: $\Lambda = \mathbb{Q}[e_1, e_2, e_3, \ldots]$ i.e. elementary symm. fns e_i are algebraic generators of Λ - every symm. fn. is a sum of products of e_i's times coefficients in $\mathbb{Q}$ (equiv. to $\{e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}\}$ being a basis).

Therefore:

Prop: A map $f: \Lambda \to \Lambda$ that respects the algebra structure ($f(a+b) = f(a) + f(b)$, $f(ab) = f(a)f(b)$, $f(c \cdot a) = c f(a)$, $c \in \mathbb{Q}$) is uniquely determined by where it sends each e_i.

Def: $\omega: \Lambda \to \Lambda$ by

$$e_d \mapsto h_d \quad \text{for all } d$$

elementary homogeneous.

Ex: $\omega(3e_{(2,1)} + 2e_{(3)}) = \omega(3e_2e_1 + 2e_3) = 3h_2h_1 + 2h_3$
$= 3h_{(2,1)} + 2h_{(3)}$.

Ex: $\omega(p_2) = \omega(x_1^2 + x_2^2 + x_3^2 + \cdots) = \omega(e_1^2 - 2e_2)$
$= h_1^2 - 2h_2 = (x_1 + x_2 + \cdots)^2 - 2(x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots)$
$= -(x_1^2 + x_2^2 + \cdots)$
$= -p_2$ (not a coincidence - as we'll see soon!)

Thm: ω is an involution: $\omega(\omega(s)) = s$ for any $s \in \Lambda$,
i.e. $\boxed{\omega(h_d) = e_d}$.

Pf. From last semester's final exam review:

$$H(t) := \frac{1}{(1-x_1 t)} \cdot \frac{1}{(1-x_2 t)} \cdot \frac{1}{(1-x_3 t)} \cdots = \sum_{n \geq 0} h_n t^n.$$

why? Recall geometric series expansion, apply to each ~~factor~~:

$$H(t) = (1 + x_1 t + x_1^2 t^2 + \dots)(1 + x_2 t + x_2^2 t^2 + \dots) \cdots$$

Coeff of t^n is sum of all combinations of x_i's that have total degree n $= h_n$.

Similarly:

$$E(t) := (1 + x_1 t)(1 + x_2 t) \cdots = \sum e_n t^n$$

$$\Rightarrow H(t)E(t) = 1, \quad \text{so} \quad \left(\sum h_n t^n\right)\left(\sum e_n t^n\right) = 1$$

$$\Rightarrow \text{for } n \geq 1, \quad \boxed{\sum_{i=0}^{n} (-1)^i e_i h_{n-i} = 0 \quad (*)} \quad \text{by gen. fn. mult.}$$

Apply ω: $$\sum_{i=0}^{n} (-1)^i h_i \, \omega(h_{n-i}) = 0$$

mult by $(-1)^n$: $$\sum_{i=0}^{n} (-1)^{n-i} \omega(h_{n-i}) h_i = 0$$

Reindex $i \to n-i$: $$\sum_{i=0}^{n} (-1)^i \omega(h_i) h_{n-i} = 0 \qquad (*')$$

Since $(*)$ is recursion uniquely defining e_i's in terms of h's, and $(\circledast')$ is same recursion for $\omega(h_i)$

have $\omega(h_i) = e_i$.

$\square$

Next goal: prove $\boxed{\omega\, p_d = (-1)^{d-1} p_d}$

<u>Lemma</u>: $$\exp\left(\sum_{n\geq 1} \frac{1}{n} p_n(\underline{x}) p_n(\underline{y})\right) = \prod_{i,j=1}^{\infty} \frac{1}{1-x_i y_j} =: \Omega$$

$\boxed{\begin{array}{l} \underline{x} = x_1, x_2, \ldots \\ \underline{y} = y_1, y_2, \ldots \end{array}}$

$$= \sum_{\lambda} \frac{1}{z_\lambda} p_\lambda(\underline{x})\, p_\lambda(\underline{y})$$

where $z_\lambda = 1^{m_1} m_1!\, 2^{m_2} m_2! \cdots$ with

$m_1 = \#\,1$'s in λ

$m_2 = \#\,2$'s in λ

$\vdots$

<u>Pf</u>: For first equality, take log of both sides:

$$\ln\left(\prod_{i,j} \frac{1}{1-x_i y_j}\right) = \sum_{i,j=1}^{\infty} \ln \frac{1}{1-x_i y_j}$$

(recall $\ln\left(\frac{1}{1-z}\right) = \sum_{n=1}^{\infty} \frac{1}{n} z^n$ from gen. fns.)

$$= \sum_{i,j=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{n} x_i^n y_j^n$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_i x_i^n\right)\left(\sum_j y_j^n\right)$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} p_n(x)\, p_n(y).$$

For second equality, expand exp: $\exp(z) = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \cdots$

$$\exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(x) p_n(y)\right) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\sum_{n\geq 1} \frac{1}{n} p_n(x) p_n(y)\right)^k$$

To get $p_\lambda(x) p_\lambda(y) = p_{\lambda_1}(x) p_{\lambda_2}(x) \cdots p_{\lambda_k}(x)$
$\cdot p_{\lambda_1}(y) p_{\lambda_2}(y) \cdots p_{\lambda_k}(y)$,

need $k = \#$ parts of λ, have to choose which parts of λ come from which of the k factors in $\left(\sum_{n\geq 1} \frac{1}{n} p_n(x) p_n(y)\right)^k$. If

$$\lambda = \underbrace{\lambda_1 \lambda_1 \cdots \lambda_1}_{m_{\lambda_1}} \cdots \underbrace{2 \cdots 2\, 2}_{m_2} \underbrace{1 \cdots 1 1 1}_{m_1}$$

Then there are $\binom{k}{m_1, m_2, \ldots} = \frac{k!}{m_1! m_2! \cdots}$ choices, and each comes w/ a $\frac{1}{i}$ for each i in λ.

So coeff of $p_\lambda(x) p_\lambda(y)$ is

$$\frac{1}{k!} \frac{k!}{m_1! m_2! \cdots} \cdot \frac{1}{1^{m_1}} \frac{1}{2^{m_2}} \cdots = \frac{1}{z_\lambda}. \qquad \square$$

<u>Lemma:</u> $$\exp\left(\sum_{n\geq 1} \frac{1}{n} (-1)^{n-1} p_n(x) p_n(y)\right) = \prod_{i,j=1}^{\infty} (1 + x_i y_j)$$
$$= \sum_{\lambda} \frac{(-1)^{n-\ell(\lambda)}}{z_\lambda} p_\lambda(x) p_\lambda(y)$$

(same proof).

Lemma: $\displaystyle\sum_\lambda m_\lambda(x)\, h_\lambda(y) = \prod_{i,j} \frac{1}{1-x_i y_j} = \Omega$

Pf: $\displaystyle\prod_{i,j=1}^{\infty} \frac{1}{1-x_iy_j} = \prod_{i,j} (1 + x_i y_j + x_i^2 y_j^2 + x_i^3 y_j^3 + \cdots)$

Coefficient of $x_1^3 x_2^2 x_3^2$: can get an x_1^3 from $x_1^3 y_j^3$ OR $x_1^2 y_a^2 \cdot x_1 y_b$ OR $x_1 y_a \cdot x_1 y_b \cdot x_1 y_c$.

Can get an x_2^2 from

$x_2^2 y_j^2$ OR $x_2 y_a \cdot x_2 y_b$

Can get an x_3^2 from

$x_3^2 y_j^2$ OR $x_3 y_a \cdot x_3 y_b$

$\Rightarrow$ coeff is $(y_1^3 + y_2^3 + y_3^3 + \cdots + y_1^2 y_2 + y_1^2 y_3 + \cdots + y_1 y_2 y_3 + \cdots)$
$\cdot (y_1^2 + y_2^2 + \cdots + y_1 y_2 + y_1 y_3 + \cdots)^2$

$= h_3(y)\, h_2(y)^2 = h_{(3,2,2)}(y)$

$\Rightarrow \Big(\text{Coeff of } m_\lambda(x)\Big) = h_\lambda(y)$. $\square$

Thm: $\omega\, p_\lambda = (-1)^{n-k} p_\lambda$ where $k = \#$ parts of λ.

Pf: Think of ω as acting on "y" variables:

$$\omega \sum_\lambda \frac{1}{z_\lambda} p_\lambda(x) p_\lambda(y) = \omega\, \Omega = \omega \sum_\lambda m_\lambda(x) h_\lambda(y)$$

$$= \sum_\lambda m_\lambda(x) e_\lambda(y)$$

$$= \prod (1 + x_i y_j)$$ (similar to lemma above) (Hwk)

$$= \sum_\lambda \frac{1}{z_\lambda} (-1)^{n-k_\lambda} p_\lambda(x) p_\lambda(y)$$

Conclusion follows by comparing coeffs of $p_\lambda(x)$. $\square$

$\boxed{\text{Later: we'll show } \omega s_\lambda = s_{\lambda^T} \text{ (} \lambda^T \text{: conjugate partition)}}$

Later: we'll show

$$\omega s_\lambda = s_{\lambda^T}$$

conjugate partition