- Intro: go over syllabus (20-30 min), take questions
- Tell them to have their project groups and ideas chosen by Spring Break. If they're doing a Qual (b) w/ me, they can do a solo project and have it double.

Refresher of symmetric functions:

$f(x_1, x_2, \dots)$ is symmetric if fixed under any permutation of the variables.

Symmetric function: symmetric series of finite degree

Ex: $f(x_1, x_2, x_3, \dots) = x_1^4 x_2^2 + \underbrace{x_i^4 x_j^2}_{} + \dots$ (degree 6)

$+ 3x_1 + 3x_2 + 3x_3 + \dots$

$= m_{(4,2)} + 3m_{(1)}$

Symmetric polynomial: set all but finitely many vars $= 0$ in symmetric function;

$f(x_1, \dots, x_n)$ invariant under permuting variables.

Ex: $f(x,y,z) = xy + yz + zx + 4x^2 + 4y^2 + 4z^2 = m_{(1,1)} + 4m_{(2)}$

Notation: $\Lambda_R(x_1, \dots, x_n)$ – symm polys in n vars w/ coeffs in R

$\Lambda_R(x_1, x_2, \dots)$ – symm fnc w/ coeffs in R.

($R = \mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}[q]$, etc)

Recall: bases of $\Lambda_R(x_1, x_2, \dots) = \Lambda_R$: $\lambda = (\lambda_1, \dots, \lambda_k)$ partition.

- Monomial: $m_\lambda = \sum_{i_1 \neq \dots \neq i_k} x_{i_1}^{\lambda_1} \cdots x_{i_k}^{\lambda_k}$

- Elementary: $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_k}$ where

$e_d = \sum_{i_1 < \dots < i_d} x_{i_1} \cdots x_{i_d}$

- <u>Homogeneous:</u> $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_k}$ where

$$h_d = \sum_{i_1 \le \cdots \le i_d} x_{i_1} \cdots x_{i_d}$$

- <u>Power Sum:</u> $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$ where

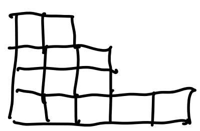
$$p_d = x_1^d + x_2^d + \cdots$$

- <u>Schur:</u> $$s_\lambda = \sum_{T \in SSYT(\lambda)} x_1^{\#1\text{'s in } T} x_2^{\#2\text{'s in } T} \cdots$$

where $SSYT(\lambda) = \{$Semistandard Young tableaux of shape $\lambda\}$:

<u>Ex:</u> $\lambda = (5,3,3,2)$

SSY Tableaux: Increasing columns
weakly increasing rows

λ: 

$$\begin{array}{|c|c|c|c|c|} \hline 5 & 5 \\ \cline{1-2} 4 & 4 & 4 \\ \cline{1-3} 2 & 3 & 3 \\ \hline 1 & 1 & 2 & 2 & 3 \\ \hline \end{array} \rightsquigarrow x_1^2 x_2^3 x_3^3 x_4^3 x_5^2$$

<u>Schur function motivation</u> (preview)

- s_λ's are characters of irreducible polynomial representations of GL_n. Multiplying Schur functions $\Leftrightarrow \otimes$, adding $\Leftrightarrow \oplus$.

- s_λ's correspond under "Frobenius map" to irred. representations of S_n.

multiplying s_λ's $\longleftrightarrow \otimes$

adding s_λ's $\longleftrightarrow \oplus$

easy things $\qquad$ hard things

- s_λ's correspond to Schubert varieties in the Grassmannian (a geometric space parameterizing k-planes in n-dimensional space),

  multiplying s_λ's $\Leftrightarrow$ intersecting (roughly)

  adding $\Leftrightarrow$ union

$\Rightarrow$ Multiplying and adding Schur functions is an all around important topic!

<u>Schur positivity</u>

<u>Def</u>: A symmetric function is <u>Schur positive</u> if, when written in the Schur basis, all its coefficients are nonnegative integers.

<u>Ex</u>: $3s_{(2,1)} + 2s_{(3)}$ is Schur positive

$3s_{(2,1)} - 0.5\, s_{(3)}$ is not.

<u>Note</u>: Schur positive $\Leftrightarrow$ Corresponds to a representation of S_n or GL_n

$$\boxed{\text{Big goal in symmetric fn theory: Proving things are Schur positive! Or generally finding combinatorial formulas for the positive integer coefficients.}}$$

<u>Ex</u>:
- e_λ, h_λ Schur positive (Hwk)
- p_λ, m_λ not Schur positive
- Product of two Schur functions is Schur positive.

"Algebraic" definition of Schur functions/polynomials

Def: A polynomial in $x_1, \ldots, x_n$ is antisymmetric if interchanging any two of the variables negates the polynomial.

i.e. for $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = \underbrace{\operatorname{sgn}(\pi)}_{\substack{\shortparallel \\ (-1)^{\mathrm{inv}(\pi)}}} \cdot f(x_1, \ldots, x_n)$$

Ex: $f(x,y) = x - y$

$$f(x,y) = x^2y - y^2x = xy(x-y)$$

$$f(x,y,z) = x^2y - y^2x - x^2z + y^2z + z^2x - z^2y$$
$$= (x-y)(x-z)(y-z)$$

Def: Given a strict partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ (λ_n could be 0) monomial antisymmetric polynomial $a_\lambda(x_1, \ldots, x_n)$ is $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_n^{\lambda_n} \pm$ similar terms

Note: If λ had any repeating parts, a_λ would be 0, since switching the corresponding x's negates the coefficient and fixes the monomial, so all monomials cancel in that case.

Ex: $a_{(4,1)}(x_1, x_2) = x_1^4 x_2 - x_1 x_2^4$

$$a_{(3,2,0)}(x_1,x_2,x_3) = x_1^3 x_2^2 - x_1^3 x_3^2 + x_2^3 x_3^2 - x_2^3 x_1^2 + x_3^3 x_1^2 - x_3^3 x_2^2$$

$\begin{array}{|c|}\hline 2\\\hline 1\\\hline\end{array}$ $\begin{array}{|c|}\hline 3\\\hline 1\\\hline\end{array}$ $\begin{array}{|c|}\hline 3\\\hline 2\\\hline\end{array}$

$$= (x_1-x_2)(x_1-x_3)(x_2-x_3)(x_1x_2+x_2x_3+x_1x_3)$$

<u>Lemma:</u> $a_\lambda(x_1,\dots,x_n)$ is always divisible by $(x_1-x_2)(x_1-x_3)\cdots(x_{n-1}-x_n) = \prod_{i<j}(x_i-x_j)$.

Moreover, for $\rho = (n-1, n-2, \dots, 2, 1, 0)$, $a_\rho = \prod_{i<j}(x_i - x_j)$.

<u>Pf:</u> Consider a_λ as a polynomial in x_1, x_2 with coefficients being polynomials in $x_3, \dots, x_n$.

For any term of the form $c x_1^a x_2^b$, we have a term $-c x_1^b x_2^a$, so we may factor:

$$c(x_1^a x_2^b - x_1^b x_2^a) = c x_1^b x_2^b (x_1^{a-b} - x_2^{a-b})$$

(if $b<a$)

$$= c x_1^b x_2^b (x_1 - x_2)(\text{polynomial})$$

Thus, factoring $x_1 - x_2$ out of terms in pairs, we see that we can factor $x_1 - x_2$ out of a_λ. Similarly we can factor $x_i - x_j$ for all $i \neq j$. Since poly. rings are UFD's, can factor all out.

The claim about a_ρ is by degree counting and checking that the leading terms match. □

<u>Note:</u> $$a_\rho = \det \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ & & \vdots & & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{i<j} (x_i - x_j)$$

is the "Vandermonde Determinant".

Why? The determinant also negates upon switching two rows. ✓

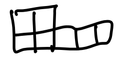
Alg. def of Schur polynomial

$$\boxed{s_\lambda = \frac{a_{\lambda+\rho}}{a_\rho}}$$

<u>Lemma:</u> $$a_\lambda = \det \begin{pmatrix} x_1^{\lambda_1} & x_1^{\lambda_2} & \cdots & x_1^{\lambda_n} \\ x_2^{\lambda_1} & x_2^{\lambda_2} & \cdots & x_2^{\lambda_n} \\ & & \vdots & \\ x_n^{\lambda_1} & x_n^{\lambda_2} & \cdots & x_n^{\lambda_n} \end{pmatrix}$$

Pf: This det is
$$\sum_{\pi \in S_n} (-1)^{inv(\pi)} x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(n)}^{\lambda_n} = a_\lambda .$$
$\square$

So the claim is the Schur function is this ratio of determinants. Pf sketch ①: show $a_\rho s_\lambda = a_{\lambda+\rho}$:

Ex: $\lambda =$ , two vars x_1, x_2:

$$\begin{array}{cccc} \begin{array}{ccccc} 2 & 2 & & & \\ 1 & 1 & 1 & 1 & 1 \end{array} & \begin{array}{ccccc} 2 & 2 & & & \\ 1 & 1 & 1 & 1 & 2 \end{array} & \begin{array}{ccccc} 2 & 2 & & & \\ 1 & 1 & 1 & 2 & 2 \end{array} & \begin{array}{ccccc} 2 & 2 & & & \\ 1 & 1 & 2 & 2 & 2 \end{array} \end{array} \Big\} \; SSYT(\lambda)$$

$$\begin{array}{cccc} \downarrow & \downarrow & \downarrow & \downarrow \\ x_1^5 x_2^2 & x_1^4 x_2^3 & x_1^3 x_2^4 & x_1^2 x_2^5 \end{array} \Big\} \; s_\lambda(x_1, x_2)$$

$$\det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} + \det\begin{pmatrix} x_1 & 1 \\ x_2 & 1 \end{pmatrix} \leftarrow a_\rho$$

$$\|$$

$$\det\begin{pmatrix} x_1^6 & x_1^5 \\ x_2^3 & x_2^2 \end{pmatrix} + \det\begin{pmatrix} x_1^5 & x_1^4 \\ x_2^4 & x_2^3 \end{pmatrix} + \det\begin{pmatrix} x_1^4 & x_1^3 \\ x_2^5 & x_2^4 \end{pmatrix} \overset{\text{want}}{=} \det\begin{pmatrix} x_1^6 & x_1^2 \\ x_2^6 & x_2^2 \end{pmatrix}$$

Full pf of this form in Eric Egge's book on symmetric functions.

Pf sketch ② (Jacobi): Relate to Jacobi-Trudi determinant formula, $s_\lambda = \det(h_{\lambda_i - i + j})$ (proven last semester).

Show $\det(x_i^{j-1}) \cdot \det(h_{\lambda_i - i + j}) = \det(x_i^{\lambda_j})$

Pf sketch ③ (cleanest but highest level):

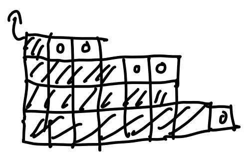
(Weyl character formula for GL_n) + (crystal base theory)

Pf sketch ④: See Stanley for theoretical pf using tools we'll develop next week.

Pf ⑤ (Proctor 1987)

Lemma: Starting w/ alg. def of s_λ, we have

$$s_\lambda(x_1,\dots,x_n) = \sum_{\substack{\mu \text{ s.t.} \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n}} s_\mu(x_1,\dots,x_{n-1})\, x_n^{|\lambda|-|\mu|}$$



← "horizontal strip"

Pf by ex: $\lambda = (4,2,1)$:

$$\frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det\begin{pmatrix} x^{4+2} & y^{4+2} & z^{4+2} \\ x^{2+1} & y^{2+1} & z^{2+1} \\ x^{1+0} & y^{1+0} & z^{1+0} \end{pmatrix}}{\det\begin{pmatrix} x^2 & y^2 & z^2 \\ x^1 & y^1 & z^1 \\ x^0 & y^0 & z^0 \end{pmatrix}}$$

set $z=1$, subtract last col from all other cols,

then divide by $x-1$, $y-1$:

$$\det\begin{pmatrix} x^5+x^4+x^3+x^2+x+1 & y^5+y^4+y^3+y^2+y+1 & 1 \\ x^2+x+1 & y^2+y+1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \Big/ \det\begin{pmatrix} x+1 & y+1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Now in each det take consecutive differences of rows:

$$\det\begin{pmatrix} x^5+x^4+x^3 & y^5+y^4+y^3 & 0 \\ x^2+x & y^2+y & 0 \\ 1 & 1 & 1 \end{pmatrix} \Big/ \det\begin{pmatrix} x & y & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} x^5+x^4+x^3 & y^5+y^4+y^3 \\ x^2+x & y^2+y \end{pmatrix} \Big/ \det\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$$

$$= \sum_{\substack{4 \geq \mu_1 \geq 2 \\ 2 \geq \mu_2 \geq 1}} \det\begin{pmatrix} x^{\mu_1+1} & y^{\mu_1+1} \\ x^{\mu_2} & y^{\mu_2} \end{pmatrix} \Big/ \det\begin{pmatrix} x & y \\ 1 & 1 \end{pmatrix}$$

which is what we want at $z=1$. Now homogenize.

QED.

<u>Cor:</u> Schur fns are a basis

<u>Pf:</u> $a_{\lambda+\rho}$'s are certainly a basis of the space of antisymm fns, which is really just $a_\rho \cdot \Lambda$. $\square$