Math 502: Combinatorics
Homework 7

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (**Mandatory** - 1 point off if you do not do this problem) Write up one or two short paragraphs describing what your final project for the class will be about and who you will be working with. See the syllabus for instructions about the final project.

2. (1+) [2 points] (2 points for all three together, not for each separately) Compute each of the following:
   (a) $s_{(5,3,3,1)} \cdot s_{(4)}$ in terms of the Schur basis
   (b) $s_{(2,1)} \cdot s_{(2,1)}$ in terms of the Schur basis
   (c) The decomposition of $V_{(2,1)} \otimes_{\text{out}} V_{(2,1)}$ into irreducible S_6 representations

3. (1+) [2 points] Use the Straightening Algorthm (Garnir Relations) to express the Garnir polynomial F_T where $T = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 1 \\ \hline\end{array}$, in terms of F_S's where each S is a standard Young tableau. Then write out each polynomial in your formula and check that your answer works.

4. (2-) [3 points] Show that
$$\mathrm{Frob}\left(\mathrm{Ind}_{S_{\lambda_1} \times S_{\lambda_2} \times \cdots \times S_{\lambda_k}}^{S_n} V_{(\lambda_1)} \otimes \cdots \otimes V_{(\lambda_k)}\right) = h_\lambda.$$

5. (2+) [4 points] Decompose the **inner** tensor product $V_{(2,1)} \otimes_{\text{inn}} V_{(2,1)}$ into irreducible representations of S_3.

   (Hint: Think of the first copy as generated by the two Garnir polynomials $x_2 - x_1$ and $x_3 - x_1$, the second as generated by the two Garnir polynomials $y_2 - y_1$ and $y_3 - y_1$. Then the inner tensor product is the vector subspace of $\mathbb{C}[x_1, x_2, x_3] \otimes \mathbb{C}[y_1, y_2, y_3] = \mathbb{C}[x_1, x_2, x_3, y_1, y_2, y_3]$ spanned by the products $(x_2 - x_1)(y_2 - y_1)$, $(x_2 - x_1)(y_2 - y_1)$, $(x_2 - x_1)(y_2 - y_1)$, and $(x_2 - x_1)(y_2 - y_1)$. Analyze how S_3 acts on these four basis elements to figure out what the decomposition must be.)

6. (1+) [2 points] Use the Murnaghan-Nakayama rule to compute $\langle s_{(3,3,1)}, p_{(3,3,1)} \rangle$.

7. (2+) [4 points] Use the Murnaghan-Nakayama rule to compute the entire character table of S_4.

8. (5) [∞ points] Let $\widetilde{H}_\lambda(x; q, t)$ be the transformed Macdonald polynomials defined in the last homework. Give a combinatorial proof, starting from the inv and maj formula, that
$$\widetilde{H}_\lambda(x; q, t) = \widetilde{H}_{\lambda^T}(x; t, q).$$
   That is, switching q, t and conjugating the partition gives the same Macdonald polynomial.