Math 502: Combinatorics
Homework 6

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (2-) [3 points] Compute the Schur expansion of the product

$$s_{3,1} \cdot s_{2,1}$$

using the Littlewood-Richardson rule. Do not use Sage (but you may check your answer on Sage in the end).

2. (2) [3 points] Prove that the insertion procedure in the proof that $c^{\rho}_{R\lambda} = c^{\lambda}_{\mu\nu}$ in the notes from Feb 20 does indeed result in a Littlewood-Richardson tableau of shape λ/μ and content ν.

3. (2) [3 points] Give a direct combinatorial proof that

$$c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$$

by giving a bijection between the Littlewood-Richardson tableaux of shape λ/μ and content ν and those of shape λ/ν and content μ. (Hint: Use JDT rectification, but also record numbers in the outer corners vacated at each step. How does this correspond to un-rectifying a highest weight tableau of content μ?)

4. (2) [3 points] Prove that the standard representation of S_n, defined as the orthogonal subspace to the all 1's vector under the permutation representation, is isomorphic to $V_{(n-1,1)}$ and is therefore irreducible.

5. (2+) [4 points] The **regular representation** of S_3 is the action by left multiplication on the vector space of formal linear combinations of elements of S_3. In matrix form, one can think of labeling the rows and columns of a 6×6 matrix by the elements of S_3, and sending each element of S_3 to the 6×6 permutation matrix given by its action by left multiplication.

   Decompose the regular representation of S_3 completely into irreducibles.

6. (3) (9 points) The n-variable **coinvariant ring** is

$$R_n = \mathbb{C}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$$

where $e_1, \ldots, e_n$ are the elementary symmetric polynomials in the variables $x_1, \ldots, x_n$. Note that S_n naturally acts on R_n by permuting the variables, since the ideal we are modding out by is S_n-invariant. Show that R_n is isomorphic to the regular representation (of S_n acting on itself, as in the previous problem).

7. **Macdonald positivity**: The **Macdonald polynomials** $\widetilde{H}_\mu(\mathbf{x}; q, t)$ are symmetric functions over the coefficient ring $\mathbb{Q}[q, t]$ defined as follows. Given a general filling σ of the boxes μ with positive integers (with no standard or semistandard condition - just literally any filling, allowing repeated letters), define $\mathrm{inv}(\sigma)$ and $\mathrm{maj}(\sigma)$ as follows. We define $\mathrm{maj}(\sigma)$ to be the sum of the major indexes of each of the columns read from top to bottom.

   For inv, a *relative inversion* of a filling σ of a Young diagram is a pair of entries u and v in the same row, with u to the left of v, such that if b is the entry directly below u (setting $b = 0$ if u is in the bottom row), one of the following conditions is satisfied:

   - $u < v$ and b is between u and v in size, in particular $u \leq b < v$.

- $u > v$ and b is not between u and v in size, in particular either $b < v < u$ or $v < u \leq b$.

Then $\mathrm{inv}(\sigma)$ is the number of relative inversions of σ.

Then

$$\widetilde{H}_\mu(\mathbf{x}; q, t) = \sum_\sigma q^{\mathrm{inv}(\sigma)} t^{\mathrm{maj}(\sigma)} x^\sigma$$

where the sum ranges over all fillings σ of shape μ and where x^σ is $x_1^{m_1} x_2^{m_2} \cdots$ where m_i is the number of i's in σ.

(a) (2-) [3 points] Compute $\widetilde{H}_{(2,1)}$ and $\widetilde{H}_{(2,2)}$. Expand them in terms of Schur functions.

(b) (2+) [4 points] Show that $\widetilde{H}_\mu(\mathbf{x}; 0, 1) = h_\mu$ and $\widetilde{H}_\mu(\mathbf{x}; 1, 0) = h_{\mu^t}$.

(c) (5+) [∞ points] Give a combinatorial proof that $\widetilde{H}_\mu(x; q, t)$ is Schur positive, that is, that the coefficient of every Schur function is a polynomial in q, t with positive integer coefficients. (This was proven by Haiman using algebraic geometry and representation theory, by finding a doubly graded S_n representation whose Frobenius character is $\widetilde{H}_\mu(x; q, t)$. No combinatorial proof is known.)