Math 502: Combinatorics
Homework 4

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

The Mandatory Problem below does not count towards your point totals for this homework.

Problems

1. **Mandatory problem (1 point off your total homework score if you don't do it):** Use the ytableau package to typeset the following skew tableau in LaTeX:

8	10							
	5	5						
		4	6	6				
		1	2	2	3			
				1	1	2	7	9

   (If you are handwriting and scanning your solutions to this homework, then handwrite the LaTeX code you used to make the tableau appear. You can use a free online LaTeX compiler like Overleaf to test your code. If you are typing up this homework in LaTeX, then you simply need to make the tableau appear on your homework pdf output.)

2. (1+) [2 points] Rectify the skew tableau shown above below using jeu de taquin.

3. (1) [1 point] Suppose λ/μ is a horizontal strip skew shape of size $|\lambda| - |\mu| = n$, consisting of rows of lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$. How many standard Young tableaux are there of shape λ/μ?

4. (1+) [2 points] Suppose λ/μ is a horizontal strip skew shape of size $n = |\lambda| - |\mu|$, consisting of rows of lengths $\alpha_1, \alpha_2, \ldots, \alpha_k$. Show that

$$s_{\lambda/\mu} = h_{\alpha_1} h_{\alpha_2} \cdots h_{\alpha_k} = \sum_{\nu} K_{\nu\alpha} s_\nu.$$

5. (1+) [2 points] Formulate and prove an analogous statement to the previous problem but for vertical strips.

6. **Chromatic symmetric functions.** Given an (undirected, labeled) graph G, a *proper coloring* of G is an assignment of a positive integer "color" to each vertex such that no two adjacent vertices have the same color. If the colors assigned to the vertices are $c_1, c_2, \ldots, c_n$ (with some c_i's possibly being equal to each other), define the *monomial* of the coloring C to be $x^C = x_{c_1} x_{c_2} \cdots x_{c_n}$. Finally, define the *chromatic symmetric function* of G to be

$$X_G(x_1, x_2, \ldots) = \sum_C x^C$$

   where the sum ranges over all proper colorings C of G.

   - (1) [1 point] Prove that $X_G(x_1, x_2, \ldots)$ is indeed a symmetric function for any graph G.
   - (1+) [2 points] Prove that, if K_n is the complete graph on n vertices, we hvae $X_{K_n}(x_1, x_2, \ldots) = n! \cdot e_n$.
   - (2-) [3 points] Compute X_G for the path graphs P_3, P_4, P_5 where P_n is the graph with n vertices. Expand them in terms of the elementary and Schur bases.

- (4) [10 points] **Shareshian-Wachs conjecture (now proven)**. A poset P is $3+1$**-avoiding** if no induced subposet is isomorphic to a disjoint union of a trivial one-element poset and a chain with three elements. Let G_P be the **incomparability graph** of a poset P, in which the vertices are the elements of P, and two elements are connected by an edge in G iff they are incomparable in P. Prove that X_{G_p} is Schur positive whenever P is $3+1$-avoiding.
- (5) [∞ points] **Stanley-Stembridge conjecture**: Prove that X_{G_p} is e-positive whenever P is $3+1$-avoiding.
- (5) [∞ points] A graph is *claw-free* if no induced subgraph among 4 of the vertices (including all edges between these four vertices) is a *claw*, which consists of a single vertex attached by edges to three others (and no other edges). Show that if G is claw-free, then X_G is Schur positive.

7. (3+) [8 points] Prove the identity

$$s_\lambda(1, q, q^2, q^3, \ldots) = \frac{\sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)}}{(1-q)(1-q^2)\cdots(1-q^n)}$$

where $n = |\lambda|$ and $\mathrm{maj}(T) = \sum_{d \text{ descent of } T} d$ and where a descent of an standard Young tableau T is an element d of T that occurs in a lower row than $d+1$. (Hint: To do this problem one can read and rewrite Stanley's proof in section 7.19.)