Math 502: Combinatorics
Homework 14

This homework is a review homework, and serves as a mini-final exam for the course. Every problem is worth 1 point and therefore you must do all of them to earn the full 10 points!

Problems

1. Consider the ring of symmetric functions $\Lambda_{\mathbb{F}_q}(x_1, x_2, x_3, \ldots)$ over the finite field $\mathbb{F}_q$. Which of the monomial, elementary, power sum, homogeneous, and Schur polynomials still bases of Λ when we are working over $\mathbb{F}_q$? Prove your answers.

2. Expand the homogeneous symmetric function $h_{3,2}$ in terms of Schur functions. (You may not use a computer; do this by hand using formulas we proved earlier in the semester.)

3. The **charge** of a permutation $w \in S_n$ is defined as follows. Label the entry 1 of w with a subscript 0. Then, for each entry $i = 2, 3, 4, \ldots, n$, if i is to the right of $i - 1$ in w then label it with one more than the subscript that labeled $i - 1$, and otherwise label it with the same subscript. Then the charge is the sum of the subscript labels.

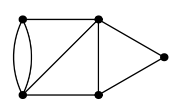
   Prove that charge is invariant under Knuth equivalence and therefore can be viewed as a statistic on the RSK insertion tableau of the permutation.

4. Show that
$$\sum_{w \in S_n} q^{\mathrm{charge}(w)} = (n)_q!$$
   where charge is as defined in the previous problem.

5. Consider the "diagonal action" of S_3 on $\mathbb{C}^6$ in which each $\pi \in S_n$ permutes the first three variables and final three variables separately (and simultaneously). Decompose this representation into irreducible representations.

6. Prove that the set of k-flats in $\mathbb{P}^n_{\mathbb{F}_q}$ form a $2 - (v, k, \lambda)$ design for some v, k, λ, and find the parameters v, k, λ in terms of n, k, q.

7. Compute the number of bases in the graphical matroid associated to the following graph (Hint: You may want to use a result discussed in 501, and you may use a computer):



8. We drew $\mathbb{P}^2_{\mathbb{F}_2}$ as the Fano plane. Draw $\mathbb{P}^2_{\mathbb{F}_3}$ in a similar manner, by drawing all of its points and all of its lines.

9. Show that the Plücker embedding $\mathrm{Gr}(n-1, n) \to \mathbb{P}^{n-1}$ is a bijection in this case.

10. Compute the product of Schubert classes $\sigma^3_{(2,1)}$ in the Chow ring $A^*(\mathrm{Gr}(3, 6))$.