(Bonus, after orbit-stabilizer and Burnside lectures)

We claimed without proof, before, that

$$\mathrm{Frob}(V_\lambda) := \sum_\mu \chi_\lambda(\mu) \frac{p_\mu}{z_\mu} = s_\lambda .$$

Steps to show this:

① Define an inner product on class functions: maps on conj. classes of S_n (to $\mathbb{C}$). Call CF_n this space,

For $f, g \in CF_n$,

$$\langle f, g \rangle = \frac{1}{n!} \sum_{\pi \in S_n} f(\pi) \overline{g(\pi)}$$

$$\left(= \frac{1}{n!} \sum_{\pi \in S_n} f(\pi) g(\pi^{-1}) \right)$$

Lemma 1: $\langle \mathrm{Frob}(V), \mathrm{Frob}(W) \rangle = \langle \chi_V, \chi_W \rangle$

Hall inner product

class fn inner product.

Pf: $\langle \mathrm{Frob}(V), \mathrm{Frob}(W) \rangle = \left\langle \sum \chi_V(\lambda) \frac{p_\lambda}{z_\lambda}, \sum \chi_W(\mu) \frac{p_\mu}{z_\mu} \right\rangle$

$$= \sum_\lambda \chi_V(\lambda) \chi_W(\lambda) \cdot \frac{1}{z_\lambda}$$

$$= \frac{1}{n!} \sum_\pi \chi_V(\pi) \chi_W(\pi^{-1})$$

same cycle type as π. □

(2) Define product structure on CF:

<u>Def</u>: $f \in CF_n$, $g \in CF_m$:

$$f \circ g = \mathrm{Ind}\uparrow_{S_n \times S_m}^{S_{n+m}} f \times g \quad \text{if } f, g \text{ characters,}$$

extend by linearity to all class functions.

<u>Lemma</u>: $\mathrm{Frob}(f \circ g) = \mathrm{Frob}(f)\,\mathrm{Frob}(g)$

<u>Pf</u>: $\mathrm{Frob}(\mathrm{Ind}\uparrow_{S_n \times S_m}^{S_{n+m}} f \times g) = \langle \mathrm{Ind}\uparrow_{S_n \times S_m}^{S_{n+m}} f \times g, \Psi \rangle$

returns $p_{c(w)}$ for any $w \in S_n$

"Frobenius Reciprocity"

$$= \langle f \times g, \mathrm{Res}\downarrow_{S_n \times S_m}^{S_{n+m}} \Psi \rangle_{S_n \times S_m}$$

$$= \frac{1}{m!\,n!} \sum_{u \in S_n} \sum_{v \in S_m} f(u) g(v) \underbrace{\Psi(uv)}_{= \Psi(u)\Psi(v)}$$

$$= \langle f, \Psi \rangle_{S_m} \langle g, \Psi \rangle_{S_n}$$

$$= \mathrm{Frob}(f)\,\mathrm{Frob}(g) \quad \square$$

R = ring of <u>virtual</u> <u>chars</u> - differences of two characters

(3) Now have ring hom. $\mathrm{Frob}: R \to \Lambda$ preserving $\langle , \rangle$'s, want to show it sends χ_λ to s_λ. Have:

- $\langle \chi_\lambda, \chi_\mu \rangle = \delta_{\lambda\mu}$ (by generic rep thy)
- $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$

Since $\{s_\lambda\}$ orthonormal $\mathbb{Z}$-basis of $\Lambda_\mathbb{Z}$, if we can show image of Frob on R is all of $\Lambda_\mathbb{Z}$, then $\mathrm{Frob}: R \to \Lambda_\mathbb{Z}$ ring isomorphism, so χ_λ's map to s_λ's (up to sign, which we can check is positive).

$\Rightarrow$ Suffices to show

$$\mathrm{Frob}: R \xrightarrow{\cong} \Lambda_\mathbb{Z}.$$

④ <u>Induction and Burnside's:</u>

Note: $\langle \mathrm{Ind}\uparrow_H^G 1_H, 1_G \rangle = \frac{1}{|G|} \sum_{w \in G} \underbrace{\# \mathrm{Fix}_{G/H}(w)}_{= \mathrm{tr}(w \text{ acting on } \mathbb{C}G/H)}$

since $\mathrm{ind}_H^G 1_H = \mathbb{C}G \otimes_H 1_H$ ← basis is given by $G/H \otimes 1$, action on G/H cosets by $w \in G$.

So Burnside's says

$$\langle \mathrm{Ind}\uparrow_H^G, 1_G \rangle = \# \text{ orbits of } G \curvearrowright G/H.$$

Use this understanding on Young subgroups:

⑤ <u>Claim:</u> $\mathrm{Frob}\left(\mathrm{Ind}\uparrow_{S_{\alpha_1}\times\cdots\times S_{\alpha_m}}^{S_n} 1_{S_{\alpha_1}\times\cdots\times S_{\alpha_m}}\right) = h_\alpha$

<u>Pf:</u> Cosets of $S_{\alpha_1}\times\cdots\times S_{\alpha_m}$ in S_n are words of length n, content α.

S_n acts by permuting the entries

<u>Ex</u>: For $\mathrm{Ind}\uparrow^{S}_{S_3 \times S_3 \times S_2} 1_{S_3 \times S_3 \times S_2}$, basis is

11122233,

11122323,

$\vdots$,

13221132, ...

$(12)\, 13221132 = 31221132$

We have $\mathrm{Ind} = 1_{S_{\alpha_1}} \circ 1_{S_{\alpha_2}} \circ \cdots \circ 1_{S_{\alpha_n}}$

and $\mathrm{Frob}\, 1_{S_m} = \sum_{\lambda \vdash m} \frac{p_\lambda}{z_\lambda} = h_m$ (proven before, or by Murnaghan-Nakayama)

Conclusion follows. $\square$

⑥ Define η^α = char of $\mathrm{Ind}\uparrow^{S_n}_{S_\lambda} 1_{S_\lambda}$. Define the virtual character

$$\varphi^\lambda = \det\left(\eta^{\lambda_i - i + j}\right)$$

Jacobi-trudi tells us

$$\mathrm{Frob}(\varphi^\lambda) = s_\lambda$$

So all s_λ's are in image $\Rightarrow \mathrm{Frob}: R \xrightarrow{\cong} \Lambda_{\mathbb{Z}}$.

$\square$