The lattice of flats

Recall: If $M = (E, \mathcal{I})$ is a matroid and $X \subseteq E$, the closure of X is

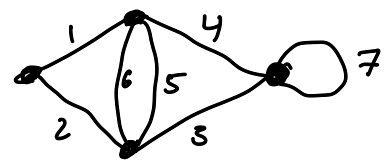
$$cl(X) = \{x \in E : rk(x \cup X) = rk(X)\}$$

(where $rk(X)$ is the rank of $M|X$).

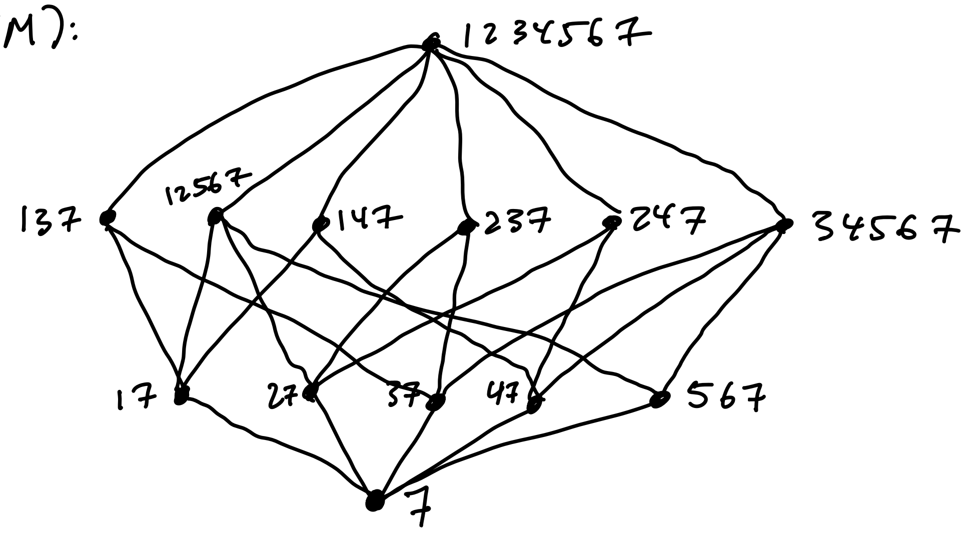
Recall: A flat is a subset $X \subseteq E$ s.t. $cl(X) = X$.

Def: The lattice of flats $\mathcal{L}(M)$ is the poset of flats of M ordered by inclusion.

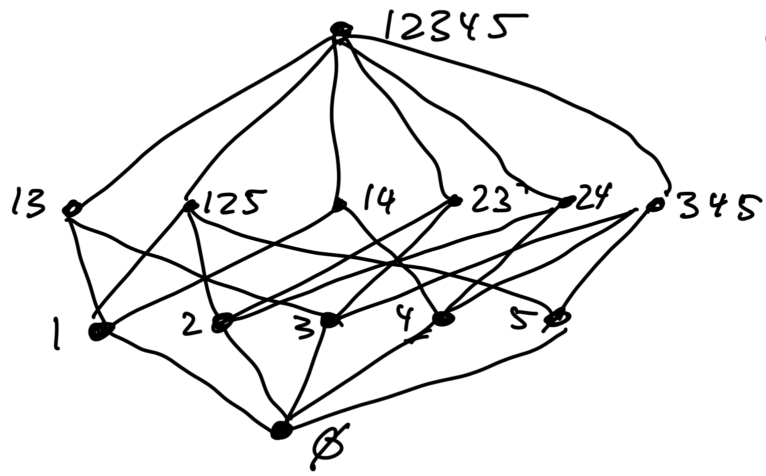
Ex: $M =$ graphical matroid:

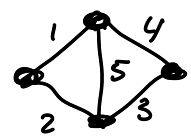


$\mathcal{L}(M)$:



Notice: If we replace G w/ corresponding simple graph, lattice is the same:





Def: A simple matroid is one with no circuits of size 1 or 2.
If M is a matroid, M_{sim} is formed by removing all elts of E that are 1-circuits and choosing one of each 2-circuit.
This leaves all other circuits unchanged; M_{sim} is a matroid (check C1–C3 for homework) and

$$\mathcal{L}(M) = \mathcal{L}(M_{sim}).$$

Facts about the lattice of flats:

Thm: If M_1, M_2 are simple matroids and $\mathcal{L}(M_1) = \mathcal{L}(M_2)$, then $M_1 = M_2$.

Thm: A finite poset is the lattice of flats of a matroid iff it is a <u>geometric lattice</u>.

Cor: Finite Geometric lattices $\Longleftrightarrow$ Simple matroids

Recall: A <u>lattice</u> is a poset s.t. every pair of elts has a <u>meet</u> (greatest lower bound, $\wedge$) and a <u>join</u> (least upper bound, $\vee$). In particular a finite lattice has a $\hat{0}$ and a $\hat{1}$.

A finite lattice is <u>geometric</u> iff it is:

- Graded (all max'l chains have same length)
- Atomic (every elt is the join of <u>atoms</u> - elts that cover $\hat{0}$)
- Upper semimodular:
$$rk(x \vee y) + rk(x \wedge y) \leq rk(x) + rk(y)$$

Ex: In example lattice above, set $x = 24$, $y = 13$.

Then $rk(x \wedge y) = rk(\emptyset) = 0$

$rk(x \vee y) = rk(12345) = 3$

$rk(x) = rk(y) = 2$

$2 + 2 > 0 + 3$. ✓

Note: $X \vee Y = cl(X \cup Y)$

$X \wedge Y = X \cap Y$

(HWK)

Pf that $\mathcal{L}(M)$ is geometric:

- Graded by rk
- Atomic b/c flats of rk 1 generate. Let X be a flat and $I \subseteq X$ max'l indep set. Then $cl(i)$ for $i \in I$ are all subsets of X since $cl(X) = X$, and certainly
  $$X = cl(I) = cl\Big(\bigcup_{i \in I} cl(i)\Big) = \bigvee_{i \in I} cl(i). \quad \checkmark$$
- Semimodular by rank axioms.

Application to geometric lattices (assuming all facts above)

Thm: Finite geometric lattices are coatomic - every elt is the meet of coatoms (covered by $\hat{1}$).

Pf: Let $\mathcal{L}$ be a finite geometric lattice and M a matroid s.t. $\mathcal{L} = \mathcal{L}(M)$.

Let X be a flat of co-rank k:
$$rk(X) = rk(M) - k.$$

We show X is an intersection of k hyperplanes (coatoms in $\mathcal{L}(M)$) by induction on k.

- $k=1$: X is a hyperplane $\checkmark$
- Assume true for $k=n$, suppose X has corank $n+1$. M has elt y not in X; $Y = cl(X \cup y)$ is a flat w/ $\begin{array}{c} Y \\ | \\ X \end{array}$, and

$Y = H_1 \cap H_2 \cap \cdots \cap H_n$ for some hyperplanes $H_1, \ldots, H_n$.

Claim: $\exists$ hyperplane H_{n+1} containing X and contained in $E(M) - y$: consider maximal closed set containing X but not y. ✓

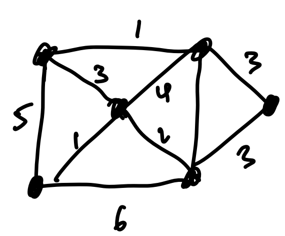
Now $y \in Y$ but $y \notin H_1 \cap H_2 \cap \cdots \cap H_{n+1} = Y \cap H_{n+1}$

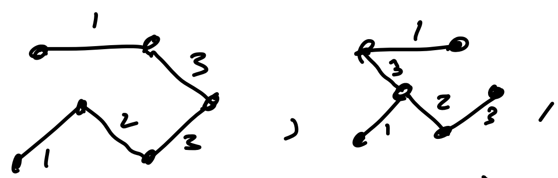
and $Y \cap H_{n+1}$ contains X (but is $< Y$)

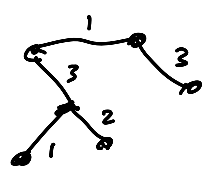
so $Y \cap H_{n+1} = X$ as desired. QED

Greedy algorithm:

Kruskal's algorithm for minimal spanning tree of edge-weighted graph: Greedy algorithm.







any application of greedy finds a minimal spanning tree.

Lemma: Let (E, I) be a matroid and $w: E \to \mathbb{R}$ be any weight function.
Then applying the greedy algorithm to find a basis of minimal weight does find a basis

of minimal weight.

Pf (Hwk). (Hint: Form an independent set at each step, use (I3) to show you can always augment in a way that is minimal)

Thm: $(E, \mathcal{I})$ is a matroid iff:

(I1) $\emptyset \in \mathcal{I}$

(I2) $\mathcal{I}$ is downward closed under $\subseteq$

(G) For all weight functions $wt: E \to \mathbb{R}$, the greedy algorithm finds a maximal member of $\mathcal{I}$ w/ maximal weight. (equiv to min by negating)

Pf: $(\Rightarrow)$ done above.

$(\Leftarrow)$ Assume I1, I2, G hold. We wish to show I3 holds. Assume ~~for~~ $\to\leftarrow$ that $\exists\, I_1, I_2 \in \mathcal{I}$ w/ $|I_1| < |I_2|$ and **no** $e \in I_2 \setminus I_1$ s.t. $I_1 \cup e \in \mathcal{I}$.

Since $|I_1 - I_2| < |I_2 - I_1|$, choose $\varepsilon < 1$ s.t.

$$\frac{|I_1 - I_2|}{|I_2 - I_1|} < \varepsilon .$$

Define

$$wt(e) = \begin{cases} 1 & e \in I_1 \\ \varepsilon & e \in I_2 - I_1 \\ 0 & \text{else} \end{cases}$$

Maximal greedy algorithm will first pick all elts of I_1. By assumption it can't pick any $e \in I_2 - I_1$, so total weight of a maximal basis is $|I_1|$.

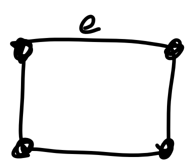
But by (I2), $I_2 \subseteq \underset{\uparrow \text{maximal}}{B_2}$ and

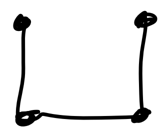
$$\begin{aligned} wt(B_2) \geq wt(I_2) &= |I_1 \cap I_2| + |I_2 - I_1|\varepsilon \\ &> |I_1 \cap I_2| + |I_1 - I_2| \\ &= |I_1| \end{aligned}$$

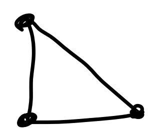
and we have a contradiction. QED.

Deletion and contraction (matroid minors)

Recall: edge deletion / contraction in graph G:

$G =$ 

$G \setminus e =$ (deletion) 

$G / e =$ (contraction) 

<u>Matroid deletion</u>: $M \setminus e = M|(E-e)$

<u>Matroid contraction</u>: $M/e = (M^* \setminus e)^*$

where $*$ is matroid <u>duality</u>.

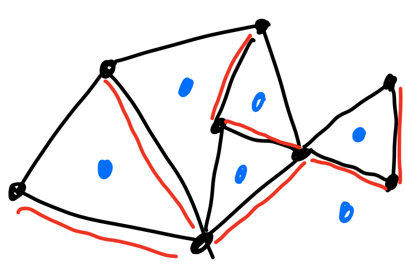
<u>Dual matroid</u>: If $M = (E, \mathcal{B})$,

$$M^* = (E, \{E-B : B \in \mathcal{B}\})$$

(Hwk: Prove that M^* is a matroid).

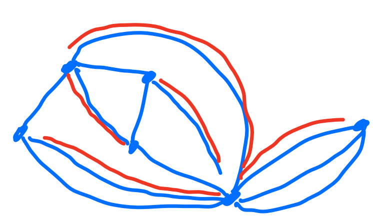
Example of a dual matroid:

$G =$ planar graph



dual graph

G^*



in a spanning tree, exactly one edge from each region is not used. $\longrightarrow$ The complement is therefore a spanning tree in the dual graph.

<u>Notice</u>: If we dualize and delete an edge, this contracts the corresponding edge in the original graph.

<u>Also note</u>: $(M^*)^* = M$.

<u>Characteristic polynomial</u> of a matroid: generalizes <u>chromatic polynomial</u> for graphs:

<u>Def</u>: G-graph. <u>Proper coloring</u> - coloring of vertices s.t. no two adjacent vertices have the same color.

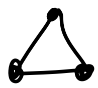
<u>Def</u>: <u>Chrom. poly</u> $\chi_G(q) =$ # proper colorings w/ q colors

<u>Ex</u>: $G =$ 

q colors $\Rightarrow$ $q(q-1)$ colorings

$$\chi_G(q) = q^2 - q$$

Ex: $G =$  $\qquad \chi_G = q^2$

Ex: $G =$  $\qquad \chi_G = q(q-1)(q-2)$

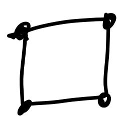
Ex: $G =$  $\qquad \chi_G = q(q-1)^2$

Recursion: $\chi_G(q) = \chi_{G\setminus e}(q) - \chi_{G/e}(q)$

distinct colors

delete an edge: now those vertices v, w can be same color

Merge v, w: can only be one color

Ex: $G =$ 

$$\chi_G = \chi_{\bullet\!-\!\bullet\!-\!\bullet\!-\!\bullet} - \chi_{\triangle}$$

$$= \chi_{\bullet}\,\chi_{\bullet\!-\!\bullet\!-\!\bullet} - \chi_{\bullet\!-\!\bullet\!-\!\bullet} - \chi_{\triangle}$$

$$= q(q)(q-1)^2 - q(q-1)^2 - q(q-1)(q-2)$$

$$= q(q-1)(q^2 - q - q + 1 - q + 2)$$

$$= (q^2 - q)(q^2 - 3q + 3)$$

$$= q^4 - 3q^3 + 3q^2 - q^3 + 3q^2 - 3q$$

$$= q^4 - 4q^3 + 6q^2 - 3q$$

Def: A sequence $(a_1, \ldots, a_n)$ is log concave if $a_i^2 \geq a_{i-1} a_i$ for all $i = 2, \ldots, n-1$.

Notice: absolute values of chrom. poly above work:

1 4 6 3

$4^2 \geq 1 \cdot 6$
$6^2 \geq 4 \cdot 3$ ✓

Def: Char. poly of a matroid is defined by

$\chi_{\circ} = 0$, $\chi_{\bullet\!-\!\bullet} = q(q-1)$, and

$$\chi_M = \chi_{M \setminus e} - \chi_{M/e}.$$

Thm: (Huh). The absolute values of the coefficients of $\chi_M(q)$ form a log-concave sequence.

→ Connections to Hodge theory
→ Tropical geometry (take Renzo's topics course in Fall '23!)