Starting from:

$$\langle s_{\lambda/\mu}, s_\nu \rangle = c^\lambda_{\mu\nu}$$

How do we get to

$$\langle s_\lambda, s_\mu \cdot s_\nu \rangle = c^\lambda_{\mu\nu} ?$$

<u>Lemma</u>: $\langle s_{\lambda/\mu}, f \rangle = \langle s_\lambda, s_\mu \cdot f \rangle$ for any symmetric function f.

<u>Pf</u>: Writing f in terms of the homogeneous basis and using the bilinearity of the Hall inner product, it suffices to prove this for $f = h_\nu$, a homogeneous symmetric function.

We have

$$\langle s_{\lambda/\mu}, h_\nu \rangle = \text{coefficient of } m_\nu \text{ in } s_{\lambda/\mu}$$
$$= \text{coeff of } x_1^{\nu_1} x_2^{\nu_2} \cdots \text{ in } s_{\lambda/\mu}$$
$$= \#\text{SSYT's of shape } \lambda/\mu \text{ and content } \nu.$$

On the other hand, consider the product $s_\mu \cdot h_r = s_\mu \cdot h_{r_1} \cdot h_{r_2} \cdots \cdot h_{r_k}$.

By the Pieri rule,

$$s_\mu \cdot h_{r_1} = \sum_{\substack{\rho/\mu \\ \text{horz strip} \\ \text{size } r_1}} s_\rho$$

so

$$s_\mu \cdot h_{r_1} \cdot h_{r_2} = \sum_{\substack{\rho/\mu \\ \text{horz} \\ \text{strip size } r_1}} s_\rho \cdot h_{r_2}$$

$$= \sum_{\substack{\rho/\mu \\ \text{horz strip} \\ \text{size } r_1}} \sum_{\substack{\lambda/\rho \\ \text{horz} \\ \text{strip size} \\ r_2}} s_\lambda$$

fill w/ 1's   fill w/ 2's

$$= \sum_{\lambda} \sum_{\substack{T \in SSYT(\lambda/\mu) \\ \text{content } (r_1, r_2)}} s_\lambda$$

Continuing this process, we find

$$s_\mu \cdot h_{r_1} \cdot h_{r_2} \cdots h_{r_k} = \sum_\lambda \sum_{\substack{T \in SSYT(\lambda/\mu) \\ \text{content } r}} s_\lambda$$

so $\langle s_\lambda, s_\mu \cdot h_r \rangle = \#$ SSYT's of shape λ/μ and content r.

The result follows. $\square$