Promotion and Evacuation

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Abstract

In this paper we take Richard Stanley's 2008 paper, Promotion and Evacuation, and go into detail of the processes and in depth with a few examples. We also state and prove a few results from the paper.

1 Introduction

Promotion and evacuation are bijections on the set of linear extensions of a finite poset. Evacuation first came from studying the RSK algorithm. Later, M.-P. Shützenberger defined evacuation without mentioning the RSK algorithm. He then extended the definition to do evacuation on linear extensions of any finite poset. Evacuation is defined in terms of a smaller operation called promotion. Shützenberger established fundamental properties of promotion and evacuation which we will explore.

All figures are printed at the back of the document for easy comparison.

1.1 Notation/Definitions

Let us first state a few definitions.

Definition 1.1. A **poset** is a partial ordering on a set P of a binary relation on P, denoted $\leq$, such that $\forall s_1, s_2, s_3$

- $s_1 \leq s_1$
- If $s_1 \leq s_2$ and $s_2 \leq s_1$ then $s_1 = s_2$
- If $s_1 \leq s_2$ and $s_2 \leq s_3$ then $s_1 \leq s_3$.

Example 1.2. Observe the set of the divisors of 30 with the binary relation divides, denoted $(30, |)$. This is a poset. Another example is the power set of any set with the relation of containment such as $(P(S_n), \subset)$.

Definition 1.3. A great way to visualize posets is called a **Hasse Diagram**. It is a graph with implied upwards orientation where vertices are the elements and two elements are connected by an edge if and only if the relation is satisfied between them.

Example 1.4. We can see the Hasse Diagram for $(30, |)$ in Figure 1.

Definition 1.5. Within a poset, $s \in P$ **covers** $t \in P$ if $t < s$ and no $u \in P$ satisfies $t < u < s$. We denote this $t \lessdot s$.

Example 1.6. Following our previous example, $3 \lessdot 15$ and $5 \lessdot 15$. Note that 30 does not cover 3 as 15 is "in the way". We can see this easily in Figure 1.

Definition 1.7. A **linear extension** is a bijection $f : P \to [p] = \{1, 2, \ldots, p\}$ such that if $s < t$ in P, then $f(s) < f(t)$. We denote the set of all linear extensions as $\mathcal{L}(P)$.

We will call the original elements of the poset "elements" and the elements of the linear extension the "labels" for clarity.

Example 1.8. A linear extension will respect all of the covering relationships from before. Let our poset be $(30, |)$. An example of a linear extension of $(30, |)$ is $f(1) = 1, f(2) = 5, f(3) = 3, f(4) = 2, f(5) = 6, f(6) = 10, f(7) = 15, f(8) = 30$. We can relabel the Hasse Diagram in Figure 1 with its linear extension as seen in Figure 2.

2 Defining Promotion and Evacuation

2.1 Promotion

Define a bijection $\partial : \mathcal{L}(P) \to \mathcal{L}(P)$ as such: Remove the label 1 say at position t_1. Take the lowest label covering t_1 and "slide" that label down to t_1 (to make it no longer empty). Continue filling the newly empty label until we have an empty space not covered by any elements. Label the last empty space $p+1$. Subtract 1 from each label to return to $1, \ldots p$ instead of $2, \ldots (p+1)$. We call this **promotion**. After applying promotion to a linear extension, we denote it $f\partial$.

Example 2.1. Observe Figure 3, promotion on a linear extension of the divisors of 30.

We naturally define **dual promotion** by removing the largest label then slide the largest label covered by it to that position. This is also the inverse of promotion. We denote dual promotion $\partial^* = \partial^{-1}$.

2.2 Evacuation

Define a bijection $\epsilon : \mathcal{L}(P) \to \mathcal{L}(P)$ by computing $f\partial$, "freezing" label p into place, and then applying ∂ to what remains and again freeze the label, $p-1$ this time. Continue until every label has been frozen. This entire process is called **evacuation**. After applying evacuation to a linear extension, denote it $f\epsilon$.

Example 2.2. Observe Figure 4.

We define **dual evacuation** as $f^* \in \mathcal{L}(P^*)$ by $f^*(t) = p + 1 - f(t)$. Then $f\epsilon^* = (f^*\epsilon)^*$. We do not go deeply into this in this paper.

NOTE: We may use promotion and evacuation on standard Young tableaux since every SYT of shape λ can be identified with a P_λ. This is how the ideas of promotion and evacuation started.

2.3 Properties of Promotion and Evacuation

Theorem 2.3. *Let P be a p-element poset. Then the operations of evacuation (ϵ), dual evacuation, (ϵ^*), and promotion (∂) have the following properties:*

(a) $\epsilon^2 = 1$ *i.e. evacuation is an involution*

(b) $\partial^p = \epsilon\epsilon^*$

(c) $\partial\epsilon = \epsilon\partial^{-1}$

3 Algebraic Interpretation

We now present a proof of Theorem 2.3 due to Haiman (and further Malvenuto and Reutenauer) that views linear extensions as words rather than functions and describes ∂, ϵ as actions on those words. We start abstractly by defining a group with generators.

Let G be the group with generators $\tau_1, \dots, \tau_{p-1}$ and relations

$$\tau_i^2 = 1 \text{ for } 1 \leq i \leq p-1$$
$$\tau_i \tau_j = \tau_j \tau_i \text{ if } |i-j| > 1.$$

Define the following elements of G:

$$\delta = \tau_1 \tau_2 \dots \tau_{p-1}$$
$$\gamma = \gamma_p = \tau_1 \tau_2 \dots \tau_{p-1} \cdot \tau_1 \tau_2 \dots \tau_{p-2} \cdot \dots \cdot \tau_1 \tau_2 \cdot \tau_1$$
$$\gamma^* = \tau_{p-1} \tau_{p-2} \dots \tau_1 \cdot \tau_{p-1} \tau_{p-2} \dots \tau_2 \cdot \dots \cdot \tau_{p-1} \tau_{p-2} \cdot \tau_{p-1}.$$

Lemma 3.1. *In the group G, we have the following identities:*

(a) $\gamma^2 = (\gamma^*)^2 = 1$

(b) $\delta^p = \gamma \gamma^*$

(c) $\delta \gamma = \gamma \delta^{-1}$

Proof. We will prove (a) by induction on p. The proofs for (b) and (c) are similar. For $p = 2$,

$$\begin{aligned} \gamma_2^2 &= \tau_1^2 && \text{by definition of } \gamma_p \\ &= 1 && \text{by generator relations.} \end{aligned}$$

Now assume $\gamma_{p-1}^2 = 1$. Then,

$$\begin{aligned} \gamma_p^2 &= \tau_1 \tau_2 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \tau_4 \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \tau_2 \cdot \tau_1 \cdot \tau_1 \tau_2 \tau_3 \tau_4 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \tau_2 \cdot \tau_1 \\ &= \tau_1 \tau_2 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \tau_4 \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \cdot \tau_3 \tau_4 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \tau_2 \cdot \tau_1 \\ &= \tau_1 \tau_2 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \tau_4 \cdot \tau_1 \tau_2 \cdot \tau_1 \tau_3 \cdot \tau_3 \tau_4 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \tau_2 \cdot \tau_1 \\ &= \tau_1 \tau_2 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \tau_4 \cdot \tau_1 \tau_2 \cdot \tau_1 \cdot \tau_4 \dots \tau_{p-1} \cdot \tau_1 \dots \tau_{p-2} \cdot \dots \cdot \tau_1 \tau_2 \tau_3 \cdot \tau_1 \tau_2 \cdot \tau_1 \end{aligned}$$

Note that all the terms inbetween the τ_4's have an index difference greater than 1 with τ_4 so we may move the two τ_4's next to each other to cancel them. This statement is then true for the τ_{i+1}'s for $i \geq 4$ since

$$\dots \tau_{i+1} \tau_1 \dots \tau_{i-1} \tau_i (...) \tau_i \tau_{i+1} \dots$$

The (...) in the middle will not cancel but be a part of γ_{p-1}^2. We continue in this way until $i = p-1$ where we reach γ_{p-1}^2 which by our induction hypothesis is 1. $\square$

For clarity, observe this smaller computation assuming we know $\gamma_4^2 = 1$. We drop the τ and write the indeces only.

$$\begin{aligned}
\gamma_5^2 &= 1234 \cdot 123 \cdot 12 \cdot 1 \cdot 1234 \cdot 123 \cdot 12 \cdot 1 \\
&= 1234 \cdot 123 \cdot 1 \cdot 34 \cdot 123 \cdot 12 \cdot 1 \\
&= 1234 \cdot 12 \cdot 1 \cdot 4 \cdot 123 \cdot 12 \cdot 1 \\
&= 123 \cdot 12 \cdot 1 \cdot 123 \cdot 12 \cdot 1 \\
&= \gamma_4^2 \\
&= 1.
\end{aligned}$$

We can definitely see a connection between ∂, ϵ and δ, γ respectively and thus a connection between Theorem 2.3 and Lemma 3.1. Since ϵ is defined in terms of ∂, we need only show the connection between ∂ and δ to use Lemma 3.1 prove Theorem 2.3.

Proof of Theorem 2.3. Let us view a linear extension $f \in \mathcal{L}(P)$ as the word $f^{-1}(1), \dots f^{-1}(p)$. For $1 \leq i \leq p-1$, define operators $\tau_i : \mathcal{L}(P) \to \mathcal{L}(P)$ by

$$\tau_i(u_1u_2 \dots u_p) = \begin{cases} u_1u_2 \dots u_p & \text{if } u_i \text{ and } u_{i+1} \text{ are comparable in P} \\ u_1u_2 \dots u_{i+1}u_i \dots u_p & \text{otherwise} \end{cases}$$

Note that τ_i is a bijection. Also note that they satisfy the relations of the abstract generators of our earlier group, G. So now we need to show that

$$\partial = \delta := \tau_1\tau_2 \dots \tau_{p-1}.$$

We want operating on the linear extension word to have the same result as doing the "sliding" on the corresponding poset. When $f = f^{-1}(1) \dots f^{-1}(p) = u_1 \dots u_p$, then $f\delta$ is obtained as follows.

STEP 1: Let $j > 1$ be the least integer such that $u_1 < u_j$. Since f is a linear extension, $u_2 \dots u_{j-1}$ are incomparable with u_1. Move u_1 between u_{j-1} and u_j.

STEP 2: Let $k > j$ be the least integer such that $u_j < u_k$. Move u_j so it is between u_{k-1} and u_k.

STEP 3: Continue in this way until the end of the word is reached.

This process is equivalent to taking a word f and factoring it left to right into maximally long factors such that the first element in each factor is incomparable with all the other elements in that factor and then cyclically shifting each factor to the left, placing the first element of each factor at the end of each factor.

Now let us consider promotion of the same $u_1u_2 \dots u_p = f^{-1}(1) \dots f^{-1}(p)$ given as a function by $f(u_i) = i$. We know the elements $u_2, \dots, u_{j-1}$ are incomparable with u_1 and thus will have their labels reduced by 1 after promotion (they will not be part of the sliding). Then label j or u_j (the least element in the linear extension f greater that u_1) will slide to u_1 and be reduced to $j-1$. Thus $f\partial = u_2u_3 \dots u_{j-1}u_1 \dots$. Continuing this, we slide the label k of u_k down to u_j. Thus $f\delta = f\partial$. □

Let us see that $f\partial$ gives the same result as $f\delta$. Figure 3, the promotion of the linear extension of the divisors of 30, gives an order in which to read the original elements of Figure 1. This results in

$$1, 3, 2, 6, 5, 15, 10, 30.$$

Now take the word of the linear extension and perform our factoring and cyclic shifting:

$$1, 5, 3, 2, 6, 10, 15, 30 = (1)(5, 3, 2, 6)(10, 15)(30) \to (1)(3, 2, 6, 5)(15, 10)(30) = 1, 3, 2, 6, 5, 15, 10, 30.$$

4 Conclusion

There are more results regarding promotion and evacuation of linear extensions. One may find these in Stanley's original paper. Stanley also goes into *self-evacuation*, linear extensions such that $f\epsilon = f$, and *P-domino tableaux*.

5 Figures

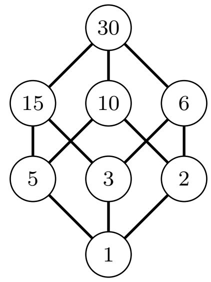

Figure 1: Hasse Diagram for the Divisors of 30

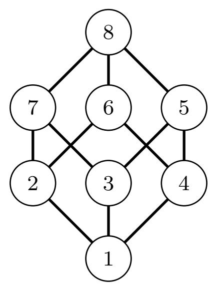

Figure 2: Hasse Diagram for the Linear Extension

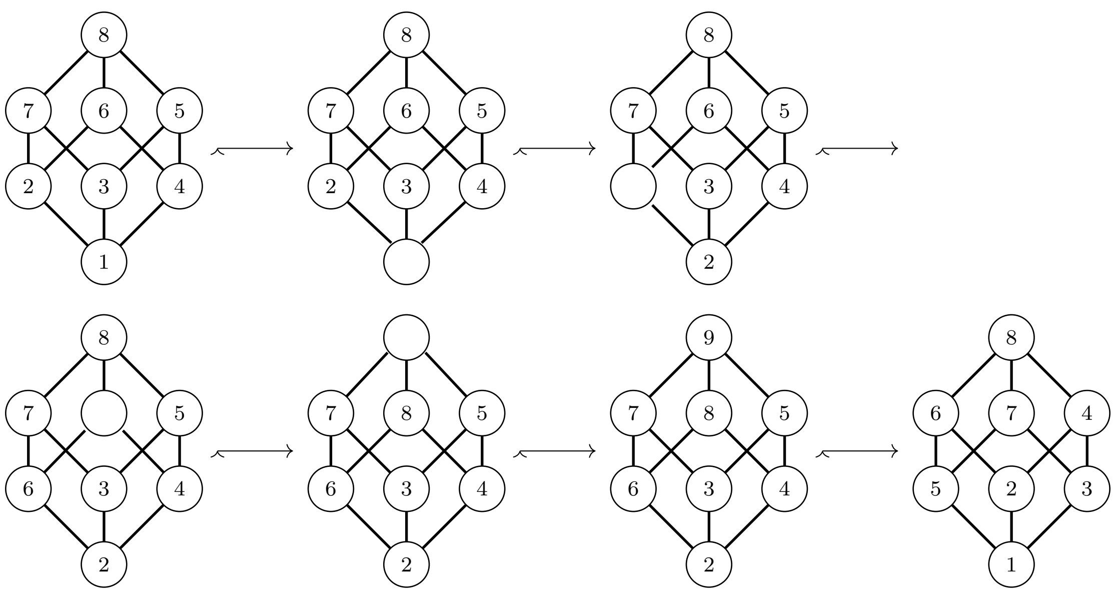

Figure 3: Promotion of Figure 2

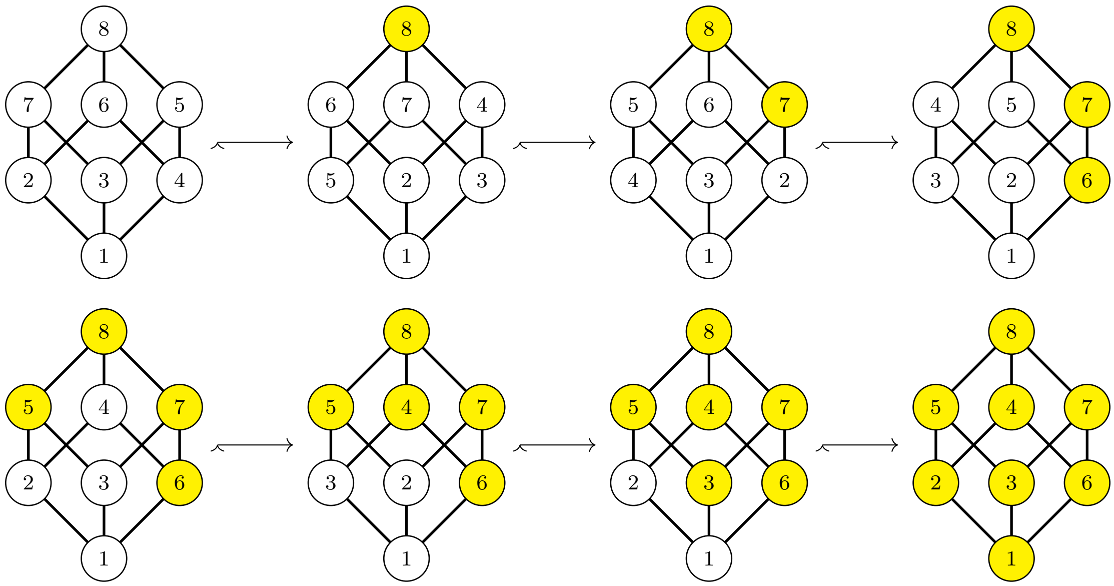

Figure 4: Evacuation of Figure 2

References

[1] R. Stanley, *Promotion and Evacuation*, The Electronic Journal of Combinatorics, (2008).