Solving Linear Intersection Problems with Schubert Varieties

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1 Background

Recall that the Grassmanian $Gr(n,k)$ is the set of all k-dimensional sub-spaces of an n-dimensional vector space, and for the purposes of this paper the ambient space will always be $\mathbb{C}^n$. An element in the grassmanian can be thought of as a $k \times n$ matrix whose rows are linearly independent. Note that row operations on this matrix leave the element in the grassmanian unchanged, so we think of the grassmanian as the set of all full-rank $k \times n$ matrices in reduced row echelon form, where the pivots in higher rows are further left than pivots in lower rows. Also recall that a full flag $F_\bullet$ in $\mathbb{C}^n$ is a set of subspaces of $\mathbb{C}^n$; $\{F_i\}_{i=0}^n$ with $F_i \subset F_{i+1}$ and $dim(F_i) = i$.

The Grassmanian $Gr(n,k)$ can be thought of as a projective variety by embedding it into $\mathbb{P}^{\binom{n}{k}-1}$ in the following manner. Given $V \in Gr(n,k)$ We order the $\binom{n}{k}$ $k \times k$ submatrices of V in some order and call them $M_{i}{}_{i=1}^{\binom{n}{k}}$, then $\varphi V = (det(M_1) : det(M_2) : \cdots : \det M_{\binom{n}{k}}) \in \mathbb{P}^{\binom{n}{k}-1}$. φ is called the Plücker embedding, and it is well-defined since row operations either change all of M_i by a common factor , or leave them all unchanged. φ is also injective so we talk interchangably talk about $Gr(n,k)$ and its image, and it so happpens that the image of φ is a variety defined by a set of quadratic equations known as the Plücker relations.

2 Schubert Cells

Any element $x \in Gr(n,k)$ has an associated partition fitting inside the partition $B = \square = ((n-k)^k)$ in the following manner. We call B the ambient rectangle. Let M be the $k \times n$ matrix in reduced row echelon form corresponding to x. Let $\lambda_k = \#$ of leading zeros in the first row, and in general $\lambda_{k-i} = -i + (\#$ of leading zeros in the $(i+1)$-th row). Hence the following matrix representing an element in $Gr(8,3)$ would have corresponding partition $(3,2,0)$.

$$\begin{bmatrix} 1 & 7 & 3 & 4 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 4 & 0 & 3 & -3 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 4 \end{bmatrix}$$

We define the Schubert cell $\Omega^\circ_\lambda = \{x \in Gr(n,k) \big| \ x \text{ has associated partition } \lambda\}$. Notice that any entry above a pivot can be cleared , and the other non-unity entries can be any complex number, so for example any element in $\Omega^\circ_{(3,2,0)}$ of $Gr(8,3)$ could be represented as

$$\begin{bmatrix} 1 & * & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

where $*$ denotes a free complex variable, and we conclude $\dim(\Omega^\circ_{(3,2,0)}) = 10$. In fact, $\dim \Omega^\circ_\lambda = k(n-k) - |\lambda|$. To see this consider the number of free variables in the $(i+1)$-th row. The $i+1$-th row corresponds to λ_{k-i} so there are $i+\lambda_{k-i}$ leading zeros , followed by a 1. There are also $k-i-1$ pivots below the i-th row, so an additional $k-i-1$ zeros.
Hence there are $n-(i+\lambda_{k-i}+1+k-i-1) = n-k-\lambda_{k-i}$ free variables. Adding over all k rows, we see dimension of $\Omega^\circ_\lambda = k(n-k) - |\lambda|$.

3 Schubert Varieties

Now that we have defined Ω°_λ we can define Schubert varieties. These are just the closure of Schubert cells in the Zariski topology, which happens to coincide with the naive idea of including limit points of Ω°_λ inside $Gr(n,k)$. It so happens that

$$\Omega_\lambda = \overline{\Omega^\circ_\lambda} = \bigsqcup_{\nu \middle| \nu \text{ contains } \lambda} \Omega^\circ_\nu$$

We will explore why this set is a variety, and why it is the smallest such containing Ω°_λ shortly, but for now we show intuitively why this occurs. We return to our example for illumination. Consider

$$\begin{bmatrix} 1 & t & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

in $\Omega^\circ_{(3,2,0)}$. Since we can rescale any row this is equivalent to

$$\begin{bmatrix} \frac{1}{t} & 1 & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

Where the $*$ entries right of the t remain $*$'s as they can scale with t to remain unchanged. Now, as we take the limit $t \to \infty$, the matrix approaches

$$\begin{bmatrix} 0 & 1 & * & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 1 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * \end{bmatrix}$$

which is an element of $\Omega^\circ_{(3,2,1)}$. The process is nearly the same for a row that is not the top row, it just involves clearing above the t before taking the limit, creating new $*$'s above the 1. You can in this manner continually add outer corners to your partition until you are at $\Omega^\circ_{((n-k)^k)}$

3.1 Schubert Varieties are Varieties

This is not just a tautology, as we have defined them so far we have no reason to suspect that Schubert varieties are in fact closed in the Zariski topology. In fact it easy to see why this is true. Since the coordinates in $\mathbb{P}^{\binom{n}{k}-1}$ are just determinants of matrices, we just need to find polynomial relations of determinants of $k \times k$ submatrices in V. But since in general a $V \in \Omega^\circ_\lambda$ has many degrees of freedom and a fixed set of 1's and 0's we can't say much beyond the fact that certain determinants vanish. In the above example of $\Omega^\circ_{(3,2,1)}$ we can see any determinant involving the first column must be zero, as well as any determinant not containing the 6-th , 7-th or 8-th columns;

and finally a determinant of the 6-th, 4-th and either 2-nd or 3-rd columns. This is the smallest variety containing $\Omega^\circ_{(3,2,1)}$. Note that as we union Schubert cells of partitions containing $(3,2,1)$ the leading 1's move to the right, and those determinants will continue to always be zero. Hence the Zariski closure precisely correponds to our naive notion of what the closure should be in the standard topology.

3.2 A note on Ω°_B

To begin with, we consider $\dim \Omega^\circ_B = k(n-k) - |B| = 0$, so Ω°_B is a collection of points. Let us examine a generic element of Ω°_B. In the i-th row, the number of leading zeros is $\lambda_{k-i+1} + i - 1 = n - k + i - 1$. Hence the leading 1 occurs in position $n-k+i$. We can then see the leading one occurs in position $n-i$ in the $k-i$-th row. What this means is there is only one element in Ω°_B , just the subspace spanned by the last k basis vectors $e_{n-k+1}, e_{n-k+2}, \ldots, e_n$.

3.3 Schubert Varieties with respect to a flag

So far we have only defined Schubert varieties by the shape of the leading 1's in the matrix representing the point of the grassmanian. We can break this statement down into something more general and then apply it to any flag. We can explicitly express Ω°_λ as

$$\Omega^\circ_\lambda = \{V \in Gr(n,k) | \dim(V \cap \langle e_1, \ldots e_r \rangle) = i \text{ for } n-k+i-\lambda_i \leq r \leq n-k+i-\lambda_{i+1}\}$$

and the corresponding Schubert variety can be seen to be

$$\Omega_\lambda = \{V \in Gr(n,k) | \dim(V \cap \langle e_n, e_{n-1}, \ldots, e_{n-\lambda_i} \rangle) \geq i\}$$

because as we take the closure, the leading 1's move to the right, and we can accumulate dimensions faster as we move backwards through the e_i.
That was a good deal of work, but it sets us up really nicely to generalize Schubert varieties to any flag. We notice that $E_\bullet$, the flag defined by $E_i = \langle e_1, \ldots, e_i \rangle$ is a valid flag (indeed a very common flag, denoted the normal flag) and that the explicit definition of the Schubert cell can be reprashed in terms of the flag. In general, for any full flag $F_\bullet$, define

$$\Omega_\lambda(F_\bullet) = \{V \in Gr(n,k) | \dim(V \cap F_{n-k+i-\lambda_i}) \geq i\}$$

as the Schubert variety with respect to the flag $F_\bullet$. Notice our original definition is the same as the Schubert variety with respect to the reverse normal flag.

4 Expressing intersection Problems in terms of Schubert Varieties

One very classical area in mathematics deals with linear intersection problems in a geometrical space. Examples of some questions you could ask in this area might be simple such as "How many lines are there between 2 points?" , "How many points do 2 lines intersect in?". They can also, however, be far more general and not immediately clear, such as "Given 4 lines in $\mathbb{R}^3$, how many lines are there that intersect all 4?"

One thing to note is that by taking the projective closure of $\mathbb{R}^n$ to $\mathbb{RP}^n$ can sometimes eliminate special cases that arise in Euclidean space. For example, consider the question "How many points do 2 lines intersect in?". If we are asking about lines in $\mathbb{R}^2$ then the answer is usually 1, unless the lines happen to be parallel. But in $\mathbb{RP}^2$, any 2 lines intersect, so the ambiguity is removed! Note, if $n > 2$ the solution is 0 points in general.

Moving into projective space also gives us an additional advantage. A k-dimensional surface in $\mathbb{P}^n$ can be viewed of as the image of a $(k+1)$-dimensional **subspace** of $\mathbb{C}^{n+1}$. This is helpful since Schubert varieties are subsets of grassmanians, and every element in a grassmanian is a subspace. This will allow us to bring to bear Schubert calculus to these intersection problems. In other words, moving into projective space and then viewing the preimage of a surface allows us to apply tools we have regarding subspaces to a surface.

For example, consider the question of determining how many lines in $\mathbb{P}^3$ intersect 4 given lines. Let $\ell_1, \ell_2, \ell_3, \ell_4$ be the 4 lines. Then when we projectivize they become the planes P_1, P_2, P_3, P_4. Let $F^i_\bullet$ be a flag where $F^i_2 = P_i$. Since we are looking for lines, we need $Gr(4, 2)$ since a line in $\mathbb{P}^3$ is the image of a plane in $\mathbb{C}^4$.

Then consider $V \in \Omega_{(1)}(F_i)$. Since $n = 4$, $k = 2$, $\lambda_1 = 1$ we have from the explicit definition of a Schubert variety with respect to a flag that $\dim(V \cap F^i_{n-k+1-\lambda_1}) \geq 1$ and $\dim(V \cap F^i_{n-k+1-\lambda_1}) \geq 2$ which simplifies to

$$\dim(V \cap F^i_2) \geq 1 \Leftrightarrow \dim(V \cap P_i) \geq 1$$

since the second statement is just that $\dim(V \cap F^i_4) \geq 2$ which is trivial. Hence V is a 2-dimensional subspace intersecting P_i with dimension at least 1, which means when we project back to $\mathbb{P}^3$, $\mathrm{im}(V)$ is a line intersecting ℓ_i in at least a point.
Now we can finally express the space of all lines intersecting each of the given 4 lines as

$$\Omega_{(1)}(F^1_\bullet) \cap \Omega_{(1)}(F^2_\bullet) \cap \Omega_{(1)}(F^3_\bullet) \cap \Omega_{(1)}(F^4_\bullet)$$

Expressing the solution set to a linear intersection problem rigorously is all well and good, but as of yet we are no closer to counting the cardinality of these sets, or even knowing if they are finite or not. To do so, we need to turn to algebraic topology.

5 Cohomology, Schur Functions, and Schubert Varieties

5.1 Cohomology and the Chow Ring

In any CW complex X , there are abelian groups $H^i(X)$ which convey information how the i-skeleton is connected to the $(i-1)$-skeleton and the $(i-2)$-skeleton. In the case of the grassmanian, the cells directly correspond to the Schubert cels (hence the name) since Ω°_B is a point, we attatch $\Omega^\circ_{(n-k)^{k-1}, n-k-1}$ which as a copy of $\mathbb{C}$ has real dimension 2, and so on; successively removing corners and adjoining until we adjoin $\Omega^\circ_\emptyset$. This is because all the Schubert cells are disjoint and isomorphic to $\mathbb{C}^{2j}$ and the maps are given via the same construction for the closure of Ω°_λ

Then the direct sum of the $H^i(X)$ is called the cohomology ring of X $H^*(X) = \oplus H^i(X)$ which is graded via H^i. The reason we care about the cohomology ring is that for the grassmanian, $H^*(X)$ has an equivalent formulation called the Chow ring, where cohomology classes in $H^*(X)$ correspond to equivalence classes of subvarieties of $Gr(n, k)$ up to birational equivalence. Birational equivalence amounts to varieties related by projective transformation, as well as degenerations in which a variety may split. In any case $\Omega_\lambda(F_\bullet)$ is birationally equivalent to $\Omega_\lambda(G_\bullet)$ for any flags $G_\bullet$ and $F_\bullet$, via some projective transformation taking the "basis" of $F_\bullet$ to the "basis" of $G_\bullet$, where basis just refers to any possible basis where the span of the first i basis elements is F_i (or G_i,

respectively).
In this correspondence, multiplication of cohomology classes corresponds to intersections of subvarieties of $Gr(n,k)$. Also $H^*(X)$ has a basis given by

$$\sigma_\lambda := [\Omega_\lambda(E_\bullet)] \in H^{2|\lambda|}(Gr(n,k)$$

where λ is a partition that fits inside the ambient rectangle B. The grading on $H^*(X)$ equates to $\sigma_\lambda * \sigma_\mu \in H^{2|\lambda|+2|\mu|}$.

5.2 Relation to Schur functions

The previous section shows the relationship between taking intersections of Schubert varieties and multiplying elements in the cohomology ring. This section will show how we relate multiplying Schur functions s_λ to multiplying homology classes. In fact, we will state without proof the following theorem.

Theorem 5.1.

$$H^*(Gr(n,k)) \cong \Delta(x_1, x_2, \dots)/(s_\lambda | \lambda \not\subset B)$$

where σ_λ corresponds to s_λ.

Now suppose $\lambda_1, \lambda_2, \dots, \lambda_r$ are partitions that fit inside the ambient rectangle and $\Sigma_i \lambda_i = k(n-k)$. Then $\sigma_{\lambda 1}\sigma_{\lambda 2}\dots\sigma_{\lambda r} \in H^{2k(n-k)}(Gr(n,k))$. There is only one generator for $H^{2k(n-k)}(Gr(n,k))$, namely σ_B. By the theorem above, the coefficient of σ_B in $\sigma_{\lambda 1}\sigma_{\lambda 2}\dots\sigma_{\lambda r}$ is the coefficient of s_B in $s_{\lambda 1}s_{\lambda 2}\dots s_{\lambda r}$. The key is we have combinatorial tools for computing the second coefficient via the Littlewood-Richardson rule. The coefficient is $c^B_{\lambda_1,\lambda_2,\dots,\lambda_r}$.
This tells us that in order to compute the cardinality of a zero dimensional intersection of Schubert varieties, we look at the corresponding Littlewood-Richardson coefficient. This is the number of points in the intersection since σ_B corresponds to a single point, the coefficient of σ_B corresponds to how many points we have, with multiplicity. We can use this to solve linear intersection problems.

6 Computing Littlewood-Richardson Coefficients

Given n and k, and r partitions $\lambda_1, \lambda_2, \dots, \lambda_r$, how do we find the Littlewood-Richardson coefficient $c^B_{\lambda_1,\lambda_2,\dots,\lambda_r}$?

The Littlewood-Richardson coefficient $c^B_{\lambda_1,\lambda_2,\dots,\lambda_r}$ is given by the number of chains of Littlewood-Richardson skew tableau $T_1, T_2, \dots, T_r$ such that the content of T_i is λ_i, each T_i extends T_{i-1} (namely $\bigcup_{j=1}^i T_j$ is a straight shape for all i), and $\bigcup_{j=1}^r T_j = B$.

6.1 Examples

We now tie everything we have learned into an example.

Suppose in $\mathbb{P}^4$, A and B are 2-planes intersecting in a point X, with $P \in A$ and $Q \in B$ be distinct points seperate from X. How many 2-planes contain P and Q and intersect each of A and B in a line?

The first step is determining what the corresponding picture looks like in $\mathbb{C}^5$. X turns into a (complex) line through the origin, with A and B turning into 3-planes, each of which has a (complex) line P or Q contained in them, where X, P, and Q are distinct. We are looking for the set S' of 3-planes S such that S intersects A on a 2-plane containing P and B on a 2-plane containing Q. This tells us we will be working in $Gr(5,3)$. Let $F_\bullet = \{P, F_2, A, F_3, F_4, F_5\}$, ie a F is a flag where the 1-dimensional subspace is P and the 3-dimensional subspace is A. Similarly let $G_\bullet = \{Q, G_2, B, G_3, G_4, G_5\}$.
Now we are looking for $V \in Gr(5,3)$ such that for a given flag $H_\bullet$ $\dim(V \cap H_1) \geq 1$ and $\dim(V \cap H_3) \geq 2$. That tells us that $n-k+1-\lambda_1 = 1$ and $n-k+2-\lambda_2 = 3$, giving us $\lambda = (2,1)$. Then we can see that

$$S' = \Omega_{(2,1)}(F_\bullet) \cap \Omega_{(2,1)}(G_\bullet)$$

Since $|(2,1)| + |(2,1)| = 6 = 3(5-3)$, the intersection is zero dimensional, and we can calculate how many elements are in the intersection by finding the Littlewood-Richardson $c^B_{(2,1),(2,1)}$. This is the number of ways to fill the partition $(2,2,2)$ with 2 skew tableaux, each with content $(2,1)$. There is only one way to fill the first one, resulting in

$$\begin{array}{|c|c|}\hline & \\ \hline \color{red}{2} & \\ \hline \color{red}{1} & \color{red}{1} \\ \hline\end{array}$$

Now we need to fill in the remaining 3 squares with a Littlewood-Richardson skew tableau with content $(2,1)$. There is only one way to do this, resulting in

$$\begin{array}{|c|c|}\hline \color{green}{1} & \color{green}{2} \\ \hline \color{red}{2} & \color{green}{1} \\ \hline \color{red}{1} & \color{red}{1} \\ \hline\end{array}$$

This tells us there is precisely one plane in $\mathbb{P}^4$ with the desired properties.

6.2 Lines through 4 given lines

Let us return to the question of finding how many lines pass through 4 given lines in $\mathbb{P}^3$. We have already shown that the set of such lines is

$$\Omega_{(1)}(F^1_\bullet) \cap \Omega_{(1)}(F^2_\bullet) \cap \Omega_{(1)}(F^3_\bullet) \cap \Omega_{(1)}(F^4_\bullet)$$

in $Gr(4,2)$. But as we have seen, this is equivalent to the number of chains of 4 tableaux filling $(2,2)$ with content (1). There are 2 such chains, illustrated below.

$$\begin{array}{|c|c|}\hline \color{yellow}{1} & \color{blue}{1} \\ \hline \color{red}{1} & \color{green}{1} \\ \hline\end{array} \text{ and } \begin{array}{|c|c|}\hline \color{green}{1} & \color{blue}{1} \\ \hline \color{red}{1} & \color{yellow}{1} \\ \hline\end{array}$$

Where the tableau are; in order, red, green, yellow, and blue. This tells us there are 2 lines through any 4 lines in $\mathbb{P}^3$

References

[1] M. Gillespie, *Variations on a Theme of Schubert Calculus*

[2] W. Fulton, *Young tableaux: with applications to representation theory and geometry*, Cambridge University Press (2012).