What's Your Strategy?

Clare Jones

May 5, 2023

Abstract

In this paper we provide an brief overview of Combinatorial Game Theory, including what it is, some basic strategies that are used in Combinatorial Game Theory, as well as some examples of these strategies in action.

1 Background

When one thinks of games, chances are what comes to mind are games like Monopoly, Settlers of Catan, and other games along those lines. However, there is a branch of games called combinatorial games. According to [1]:

Definition 1.1. In a combinatorial game there are two players who take turns moving alternately. Play continues until the player whose turn it is to move has no legal moves available. No chance devices such as dice, spinners, or card deals are involved, and each player is aware of all the dtails of the game position (or game state) at all times.

In other words, only one player moves at a time, rather than the game being played in a free for all manner, such as in SET. Also, unlike in games such as Monopoly, where in order to make a move the player must first roll the dice, in combinatorial games, players can simply move according to the rules for a given game. Finally, unlike certain card based games, where each player positions their cards in such a way that the other players cannot see what they are holding, in a combinatorial game, all of the information that the game gives is available to all players.

Combinatorial games end when players can no longer make any moves, and typically the last player to make a move wins. Studying these games, and the strategies used to win these types of games, is a field of math known as combinatorial game theory [1].

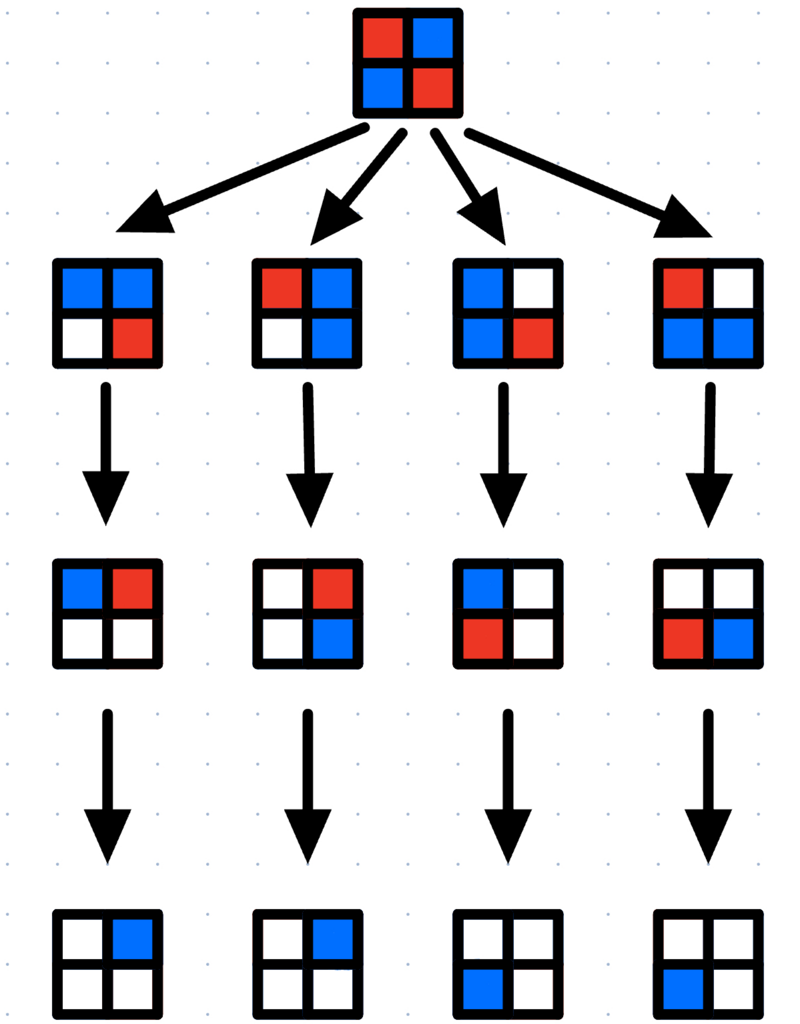
The reader may have heard of the term game theory before, just with regards to classical, or economic game theory. The main difference between economic game theory and combinatorial game theory is in economic game theory, the players make their moves at the same time and are unaware of the moves that their opponent is making [1]. So, even though each player knows what move their opponent is making in a combinatorial game, once a player makes a move, there is potentially a myriad of moves that can come after, which makes the greatest challenge in combinatorial game theory simply going through all possible moves and trying to determine the winning strategy, which is defined in [1] as follows:

Definition 1.2. A *winning strategy* in a combinatorial game is one that will guarantee a win for the player employing that strategy no matter how his or her opponent chooses to play.

So, a winning strategy will allow for the player, either the first or second player, to win regardless of the level of experience of their opponent. As for assessing all possible moves to determine if there is a winning strategy for a specific player, this is when we employ something called a game tree which is done in the following way by [1]:

- Create a node for the original position, and draw nodes for each of its options, placing them below the first node. Then draw a directed edge from the top node to its options.
- For each option, again draw nodes for each of its options, placing them below and drawing a directed edge to these nodes.
- Repeat with any subposition that still has unexpanded options.

An example of a game tree can be seen below for a game called Clobber which will be defined later:



As can be seen, regardless of which moves red or blue makes, with this coloring of the Clobber board on a 2×2 square, blue is guaranteed a win.

2 Strategies

There is no one way to win a combinatorial game, but there are a few strategies that make recurring appearances in proofs of winning strategies. We will explore those strategies now.

2.1 Greedy

We will begin by looking at the greedy strategy. According to [1], a greedy strategy is as follows:

Definition 2.1. A player following a *greedy strategy* always chooses the move that maximizes or minimizes some quantity related to the game position after the move has been made.

In other words, every time a player makes a move, they are trying to get the biggest bang for their buck. Almost every game, combinatorial or not, has aspects of the greedy strategy in them [1]. For instance, consider Monopoly, where the players are trying to get the most money and properties in order to beat their fellow players. Or think about checkers, where the players are trying to claim their opponent's pieces. An even more simple example are games such as Apples to Apples or Cards Against Humanity, where the goal is to get the most prompt cards by the end of the game. In this example, the move that the player does is trying to select the card that they believe the judge will pick to award the prompt card to, so they are maximizing their chances of winning by selecting that card.

As we will see while exploring all these strategies, the type of game where this strategy is useful has different properties. For instance, in games where the player's goal is to claim the most pieces or cover the most distance in the game, a greedy strategy may be useful. [1] even mentions games where the winner is the one with the highest score could benefit from this strategy.

However, games with the winning strategy of greedy are not the most interesting to look at, so we will spend our time looking at games with different types of winning strategies.

2.2 Symmetry

Another strategy to consider is the strategy of symmetry. This is when one of the players chooses their move in such a way that their move is either a reflection or a rotation of the move made by their opponent. This is useful because if one player can mirror their opponent's moves, then they will have a move as long as their opponent has a move.

An example of this is given in [1]:

> A famous chess wager goes as follows: An unknown chess player, Jane Pawnpusher, offers to play two games, playing the white pieces against Garry Kasparov and black against Anatoly Karpov simultaneously. She wagers one million dollars that she can win or draw against one of them. Curiously, she can win the wage without knowing about chess. How?

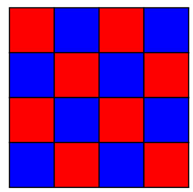
This is a prime example of symmetry, as [1] goes on to explain how she lets Karpov go first and makes that move against Kasparov, and then mirrors the move that Kasparov makes against her in her game with Karpov. This way, if Karpov wins, she is able to beat Kasparov, and vice versa. [1] calls this type of symmetry the *Tweedledum-Tweedledee* or *copycat* strategy.

An example of a game setup with this strategy as the winning strategy can be seen in a game called Clobber which is defined in [1] as follows:

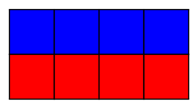
Definition 2.2. Position: A rectangular board, each square is empty or occupied by a blue or red stone. Play often begins with a board filled with alternating blue and red stones in a checkerboard pattern.

Moves: Left (player one) moves a blue stone onto an orthogonally adjacent red stone and removes the red stone. Right (player two) moves a red stone onto an orthogonally adjacent blue stone and removes the blue stone.

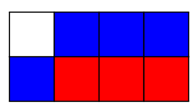
A typical Clobber board looks as follows:



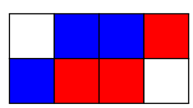
However, for our example, which is an exercise from [1], consider the following Clobber board:



This is still, by Definition [2.2], a Clobber board. This board can be won by using a symmetry strategy. For instance, suppose blue makes the following move:

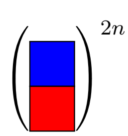


Now, red has four potential moves, but if red decides to use the symmetrical approach, red makes the following move:



By playing with symmetry of 180°, red can always mirror blue with the coloring of this board, which means that no matter which move blue makes, red will always have a move to make as well. Thus, red will always win, as the winner in a game of Clobber is the player who makes the last move.

There is a variation of this problem that was mentioned in [1]. We can generalize and rewrite the board seen above as:

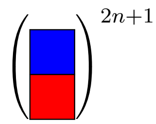


Which is proven by what was written above.

Proof. Blue makes a first move. Red makes their move using 180° symmetry. Since this board has a symmetrical coloring, as well as an even number of boxes in each row, with the 180°, Red can view the board as Blue does, with red on top and blue on the bottom. Thus, whenever Blue makes a move, Red can mirror the move, guaranteeing that whenever Blue has a move, Red does as well, thus ensuring that Red will be able to make the last move.

□

However, the same coloring with an odd exponent:



had been proven to be a first player win when $n \leq 13$ at the time of the publishing of [1], and is believed to be true for all odd n, but this has not been proven.

2.3 Parity

The next strategy is parity, which, according to [1], is defined below:

Definition 2.3. A number's *parity* is whether the number is odd or even

The trick behind parity is similar to the trick behind symmetry [1]. With symmetry, if blue is guaranteed a move regardless of the move that red makes, then blue will be the last player to make a move, which typically in combinatorial games, means that blue is the winner. In parity, the players are taking turns performing an action, where the last player to do so wins the game. Thus, since blue is going first, blue wants there to be an odd number of moves, while red, who is going second, wants there to be an even number of moves [1].

Imagine red and blue are playing a game where the goal is to take turns removing straws from a canister. In order to win, the player must be the last one to remove a straw, and each player can only remove one straw at a time. So, if there is an odd number of straws present then blue, who is going first, will be the winner. However, if there is an even number of straws, then red, who is going second, will be the winner.

This game may seem to be a rather boring one. However, there are other games of a similar vein that also use parity but appear to be a bit more complex at face value. Consider the following scenario inspired by [1]:

> Two children are fighting over how to split their Halloween candy, as they both like every piece that is present in their stash. Their mother, tired of all the arguing, and knowing some combinatorial games, decides to suggest that they play a game with a parity strategy. The children combine their candy together and will take turns splitting the large stash into smaller piles, and the last child to be able to make a move gets pick one of the pieces of candy to keep. The game will continue until there is only one piece left, where the child who goes second will win, as the child who goes first cannot make a move.

At first glance, this game may seem like a terrible way to split their candy. However, if each child puts in an equal amount of candy, say six pieces, and they go in the same order each time, they will each end up with six pieces, and this can be seen using parity.

No matter how they each split the stash of candy, if there are n pieces of candy in the pile, there are $n - 1$ moves. For instance, look below:

| 12 | → | 8 | 4 | → | 6 | 4 | 2 | → | 5 | 4 | 2 | 1 | → | 4 | 3 | 2 | 2 | 1 | →

| 3 | 2 | 2 | 2 | 2 | 1 | → | 2 | 2 | 2 | 2 | 2 | 1 | 1 | → | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 | →

2	2	2	1	1	1	1	1	1

$\rightarrow$

2	2	1	1	1	1	1	1	1	1

$\rightarrow$

2	1	1	1	1	1	1	1	1	1	1

$\rightarrow$

1	1	1	1	1	1	1	1	1	1	1	1

As the reader can see, there were only eleven moves, and this is because of the fact that by the time the last player makes their move, there will by twelve piles each of size one. So whether the players each just move one piece at a time or try to do something more random, the number of moves will always be one less than the number of pieces in the first pile. Thus, after the kids figure out who goes first, the first player gets the first piece of candy. However, once there are only eleven pieces left, then the second child gets the piece of candy, as there would be a total of ten moves, as ten is one less than eleven. This continues on until there is only one piece left, in which case the second child gets the piece of candy, as was explained above, and the mom is happy because her two kids have stopped fighting and each get the same amount of candy. Of course, now they just have to decide who gets to go first. A proof of why there are always $n-1$ moves can be seen below:

Proof. Take a stack with n items in the stack. Separate the stack into two piles. Since there has to be at least one item in a pile at all times, and the fact that we cannot just move a whole stack, there is one piece in each of those two piles that will never leave those piles. This holds true for every move that is made after.

The game ends when there are n piles of size one, so by the end of the game, we will have created $n-1$ additional stacks. Since a new stack is created each turn, and once a stack is created it must have at least one item in it, there are only $n-1$ moves that can be made. If there were less moves, then we would not have n piles of size one, as at least one pile would still contain at least two pieces. Similarly, if there were more than $n-1$ moves, then more than $n-1$ stacks would have been created, meaning we would have more than n stacks at the end, but since there are only n items, the most amount of stacks we can have is n. □

2.4 Give Them Enough Rope

The strategies mentioned up to now work in specific cases, as in the player has to go first or the player has to go second, otherwise the strategies do not apply to their case. However, what if there was not a winning strategy for one of the players, but a strategy that was fairly good, as in if player one played a certain way, very carefully, they were almost guaranteed a win, and the second player wants a chance to win but does not know a strategy that would allow for them to easily obtain victory. That is when, according to [1] the second player can employ the strategy known as give them enough rope. Essentially, the second player wants to play in such a way that the first player ends up making a mistake that costs them the victory.

In other words, if the second player can see that they are most likely going to lose, they might as well go down fighting. Their goal in this instance is to cause as much chaos as possible as to confuse their opponent to where the first player makes a mistake, which allows for the second player to win the game, regardless if they know of a winning strategy or not. As a caveat, [1] also mentions that if one of the players is already winning they should try to simplify the board as much as possible, so as to make the win easier for themselves. After all, if they can see that they are going to win, why not make their life easier and cut down the risk of them making a mistake that could cost them the game.

At the end of this section of [1] however, the authors do mention the possibility of one of the players not giving their opponent any rope. This can be done in such a way so that one player can remove some of their opponent's moves and increase their own. The authors of [1] do give a caveat though about this method potentially being a trial and error way of doing things.

2.5 Change the Game

Every once in a while, one game can appear to look like a different game. So, if game A looks like game B, and red or blue has a winning strategy for game B, then they may be able to apply that strategy to game A.

Consider the following example inspired by the example given in [1]:

> In 3-to-18, there are nine cards, face up, labeled with the digits {2,3,. . . , 10}. Players take turns selecting one card from the remaining cards. The first player who has three cards adding up to eighteen wins.

According to [1], this game is suspiciously similar to Tic Tac Toe. First of all, the players would have to construct a magic square which is defined as follows in [2]:

Definition 2.4. A *magic square* is a square array of numbers consisting of the distinct positive integers $1, 2, ..., n^2$ arranged such that the sum of the n numbers in any horizontal, vertical, or main diagonal line is always the same number

We can construct a magic square with the numbers $\{2, 3, \ldots, 10\}$ in the following way:

5	4	9
10	6	2
3	8	7

The reader can confirm that each row, column, and diagonal do in fact add up to eighteen. So now, since each player is trying to be the first to get three cards that add up to eighteen, they are also trying to be the first person to get three in a row, whether that be a row, column, or a diagonal. This allows for us to apply our knowledge of Tic Tac Toe to this game, which allows for us to conclude that this game will, more likely than not, end up in a tie.

3 Conclusion

These are just some basic strategies for combinatorial games. The strategies that are used in proofs build on these and are slightly more complicated.

One might wonder what other games these strategies could be applied to that were not included in the book. In theory, these strategies can be applied to any game does not rely on chance, as is included in the definition of a combinatorial game.

The main question is, what game shall we play next?

4 Acknowledgements

The author would like to acknowledge the following people for their help in completing this paper.

The author's mother for her support in the author following the author's dreams and her patience in putting up with all of the author's stress.

The author's dog Reine for helping the author to not focus too intensely on anything to the point where the author stresses herself out.

The author's undergraduate advisor Dan Daly for introducing the author to this field of math and helping the author to explore her mathematical interests.

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[1] Albert, M. H.; Nowakowski, R. J.; Wolfe, D. Lessons in play: An introduction to combinatorial game theory, Second edition; AK Peters, 2019.

[2] Weisstein, Eric W. "Magic Square." From MathWorld–A Wolfram Web Resource. https://mathworld.wolfram.com/MagicSquare.html