Introduced involution ω and Hall inner product $\langle , \rangle$, last week.

Recall:

- $\omega e_n = h_n$
- $\omega h_n = e_n$
- $\omega p_k = (-1)^{n-k} p_k$

Q1 $\omega m_\lambda = ?$

Q2 $\omega s_\lambda = ?$

- $\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu}$
- $\langle p_\lambda, \frac{p_\mu}{z_\mu} \rangle = \delta_{\lambda\mu}$

Q3 $\langle ??, e_\mu \rangle = \delta_{\lambda\mu}$

Q4 $\langle s_\lambda, ?? \rangle = \delta_{\lambda\mu}$

Q1: define $f_\lambda = \omega m_\lambda$. f_λ's called the "forgotten" basis, have few interesting properties.

Q3: Recall $\langle f, g \rangle = \langle \omega f, \omega g \rangle$. Thus

$$\langle \omega m_\lambda, \omega h_\mu \rangle = \delta_{\lambda\mu}$$

$$\Rightarrow \quad \langle f_\lambda, e_\mu \rangle = \delta_{\lambda\mu}$$

So f, e are dual bases.

Q2: Goal: $\omega s_\lambda = s_{\lambda^T}$ ← transpose or conjugate

Q4: Goal: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$ ← orthonormal (self-dual) basis!

Both Q2 and Q4 goals require us to prove:

Prop: $$\sum_\lambda s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \prod_{i,j=1}^{\infty} \frac{1}{(1-x_i y_j)} = \Omega = \sum m_\lambda(x) h_\lambda(y)$$

Since terms from first sum are of the form

$$x^T y^S \quad \text{where } T, S \text{ both SSYT's}$$

of shape λ, want to study the combinatorics of pairs (T, S).

RSK (Robinson-Schensted-Knuth) Bijection/Algorithm

Bijection

$$\left(\begin{array}{l} \text{Pairs of SSYT's} \\ (T,S) \text{ of same shape} \\ \lambda \vdash n \end{array} \right) \longleftrightarrow \left(\begin{array}{l} \text{two-line arrays} \\ \text{of length } n \end{array} \right)$$

where a <u>two-line array</u> is a $2 \times n$ matrix of numbers s.t. the bottom row is weakly increasing and the top row weakly increases above a constant block on bottom:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ b_1 & b_2 & \cdots & b_n \end{pmatrix}$$

s.t. $b_1 \leq b_2 \leq \cdots \leq b_n$ and if $b_i = b_{i+1}$ then $a_i \leq a_{i+1}$.

<u>Ex:</u> $\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 2 & 3 & 3 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \end{pmatrix}$

Insertion algorithm: "Insert" the top row one letter at a time to form a tableau:

- Put down first letter $\boxed{1}$
- Try to insert next letter b in first row. If it's largest, put it @ end $\boxed{1}\boxed{1}\boxed{2}$

- If not, let a be the leftmost entry of 1st row s.t. $a > b$. Replace a w/ b ("bump" a out) and then insert a in 2nd row.

$$1\,1\,2 \leftarrow 1 \qquad \begin{array}{l} 2 \\ 1\,1\,1 \end{array}$$

Meanwhile, "record" using the bottom row, keeping the same shape at each step:

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix}$$

Insertion (S)	Recording (T)
$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$	$\begin{array}{\|c\|}\hline 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|}\hline 1 & 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 2 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\hline 1 & 1 & 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|c\|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 \\ \hline\end{array}$
$\begin{array}{\|c\|c\|c\|c\|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 & 4 \\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\cline{1-1} 2 \\ \hline 1 & 1 & 1 & 2 \\ \hline\end{array}$
$\begin{array}{l} 2\,4 \\ 1\,1\,1\,2 \end{array}$	$\begin{array}{l} 2\,3 \\ 1\,1\,1\,2 \end{array}$

$$\begin{array}{lllll} 2 & 4 & & & \\ 1 & 1 & 1 & 2 & 3 \end{array} \qquad \begin{array}{lllll} 2 & 3 & & & \\ 1 & 1 & 1 & 2 & 3 \end{array}$$

$$\begin{array}{lllll} 4 & & & & \\ 2 & 2 & & & \\ 1 & 1 & 1 & 1 & 3 \end{array} \qquad \begin{array}{lllll} 4 & & & & \\ 2 & 3 & & & \\ 1 & 1 & 1 & 2 & 3 \end{array}$$

$$T = \begin{array}{|c|c|c|c|c|} \hline 4 & \multicolumn{4}{c}{} \\ \hline 2 & 2 & 3 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 & 1 & 2 \\ \hline \end{array} \qquad S = \begin{array}{|c|c|c|c|c|} \hline 4 & \multicolumn{4}{c}{} \\ \hline 2 & 3 & 4 & \multicolumn{2}{c}{} \\ \hline 1 & 1 & 1 & 2 & 3 \\ \hline \end{array}$$

Why is this a bijection?

__Lemma:__ RSK is a bijection between (Knuth)

(Pairs of __standard__ YT's of same shape) $\longleftrightarrow$ (permutations)

$$\text{Ex: } \left(\begin{array}{|c|c|c|} \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array} \, , \, \begin{array}{|c|c|c|} \hline 2 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 3 & 5 \\ \hline \end{array} \right) \longleftrightarrow \begin{pmatrix} 3 & 1 & 4 & 2 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = 31425$$

__Pf:__ For this case, start w/ biggest label in recording tableau and "unbump" that entry from insertion tableau.
i.e. remove it from the end of its row and insert into row below, bumping out largest entry __smaller__ than it, and so on.

Unbumped letters in reverse order form a permutation. $\square$

Lemma: (Schensted) Bijection

$$\left(\begin{array}{c}\text{Pairs (SSYT, SYT)} \\ \text{same shape}\end{array}\right) \longleftrightarrow (\text{words})$$

Later:

- Increasing subsequences
- Knuth equivalence
- Rectification
- Plactic Monoid

Ex:

$$\left(\begin{array}{ccc}2&3& \\ 1&1&3\end{array}, \begin{array}{ccc}2&4& \\ 1&3&5\end{array}\right) \longleftrightarrow \left(\begin{array}{ccccc}2&1&3&1&3 \\ 1&2&3&4&5\end{array}\right)$$

(Proof similar).

Def: Reading word $\text{rw}(T)$ of tableau: Concatenate rows top to bottom.

$$\text{rw}\left(\begin{array}{ccc}2&3& \\ 1&1&3\end{array}\right) = 23113$$

Reading order: The ordering of the squares given by the reading word, left to right.

Standardization: of an SSYT is the unique SYT w/ same relative order of entries, w/ ties broken in reading order:

Standardization of a word: Same thing.

$$\begin{array}{ccc}
\left(\begin{array}{ccc}2&3& \\ 1&1&3\end{array}, \begin{array}{ccc}2&4& \\ 1&3&5\end{array}\right) & \xrightarrow{\text{RSK}} & 21313 \\
\Big\downarrow \text{STD} & & \Big\downarrow \text{STD} \\
\left(\begin{array}{ccc}3&4& \\ 1&2&5\end{array}, \begin{array}{ccc}2&4& \\ 1&3&5\end{array}\right) & \xrightarrow{\text{RSK}} & 31425
\end{array}$$

Note: Given a goal content, unique way to destandardize.

Destandardize $\begin{matrix} 3 & 4 & \\ 1 & 2 & 5 \end{matrix}$ to get
- ~~two~~ two 1's
- one 2
- two 3's

$$\begin{matrix} 3 & 4 & \\ 1 & 2 & 5 \end{matrix} \xrightarrow{STD^{-1}} \begin{matrix} 2 & 3 & \\ 1 & 1 & 3 \end{matrix}$$

Pf of Schensted bijection:

$$\begin{tikzcd}
\left(\begin{array}{c}\text{Pairs (SSYT content } \mu, \text{ SYT)} \\ \text{same shape}\end{array}\right) \arrow[d, shift left, "STD"] & \left(\begin{array}{c}\text{words of} \\ \text{content } \mu\end{array}\right) \arrow[l, "RSK"'] \arrow[d, shift left, "STD"] \\
\left(\begin{array}{c}\text{Pairs of SYT's,} \\ \text{same shape,} \\ \text{size } n = |\mu|\end{array}\right) \arrow[u, shift left, "STD_\mu^{-1}", "\cong"'] \arrow[r, leftrightarrow, "RSK", "\cong"'] & \left(\begin{array}{c}\text{Permutations} \\ \text{length } n = |\mu|\end{array}\right) \arrow[u, shift left, "\cong"]
\end{tikzcd}$$

Commuting diagram $\Rightarrow$ top arrow is a bijection too.

Key Lemma: If $a \leq b$ and T is a tableau and we compute $T \leftarrow \boxed{a} \leftarrow \boxed{b}$, then the insertion path of a in T (squares involved in bumping/placing) lies strictly to the left of the insertion path of b in T.

Pf: Induct on rows. Example:

$$\begin{array}{|c|c|c|c|c|c|} \hline 4 & 4 \\ \hline 3 & 3 & 5 & 5 & 5 \\ \hline 2 & 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 1 & 2 & 2 & 3 \\ \hline \end{array} \leftarrow \boxed{1} \leftarrow \boxed{2}$$

$\square$

Now: Full RSK bijection <u>also</u> commutes with standardization, due to Key lemma

$$\begin{tikzcd}
\left(\begin{array}{|c|c|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 2 & 3 \\ \cline{1-3} 1 & 1 & 1 & 1 & 2 \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|c|c|} \hline 4 \\ \cline{1-1} 2 & 3 & 4 \\ \cline{1-3} 1 & 1 & 1 & 2 & 3 \\ \hline \end{array} \right) \arrow[d, "\text{STD right}", "\cong"'] & \begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \end{pmatrix} \arrow[l, "\text{RSK}"'] \arrow[d, "\text{STD bottom}", "\cong"'] \\
\left(\begin{array}{ccccc} 4 \\ 2 & 2 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{array} \;,\; \begin{array}{ccccc} 8 \\ 4 & 6 & 9 \\ 1 & 2 & 3 & 5 & 7 \end{array} \right) & \begin{pmatrix} 1 & 1 & 2 & 1 & 4 & 2 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{pmatrix} \arrow[l, "\cong"']
\end{tikzcd}$$

Therefore, full RSK is a bijection too.

Back to Schur function goals:

<u>Prop</u>: $\sum s_\lambda(\underline{x}) s_\lambda(\underline{y}) = \sum m_\lambda(\underline{x}) h_\lambda(\underline{y})$

<u>Pf</u>: $$\sum_\lambda s_\lambda(x) s_\lambda(y) = \sum_\lambda \Big(\sum_{T \in SSYT(\lambda)} x^T \Big) \Big(\sum_{S \in SSYT(\lambda)} y^S \Big)$$

$$= \sum_{\substack{(T,S) \text{ SSYT's} \\ \text{same shape}}} x^T y^S$$

$$= \sum_{\substack{\text{two line arrays} \\ \binom{a_1 \cdots a_n}{b_1 \cdots b_n}}} x_{a_1} x_{a_2} \cdots x_{a_n} y_{b_1} y_{b_2} \cdots y_{b_n}$$

Coefficient of $y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_k^{\lambda_k}$ is

$$\sum_{\substack{a_1, \dots a_n \\ \text{s.t. } a_1 \le a_2 \le \dots \le a_{\lambda_1}, \\ a_{\lambda_1+1} \le \dots \le a_{\lambda_1+\lambda_2}, \\ \vdots}} x_{a_1} \cdots x_{a_n}$$

$$= h_{\lambda_1} \cdot h_{\lambda_2} \cdots h_{\lambda_k}(\underline{x})$$

$$= h_\lambda(\underline{x})$$

$\Rightarrow$ Coeff of $m_\lambda(y)$ is $h_\lambda(x)$.

QED.

Cor: $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}$.

Cor 2: $\omega s_\lambda = s_{\lambda^T}$:

Pf: $\langle s_\lambda, h_\mu \rangle = K_{\lambda\mu}$ $\leftarrow$ coeff of monomial m_μ in s_λ

$\Rightarrow \langle \omega s_\lambda, e_\mu \rangle = K_{\lambda\mu}$

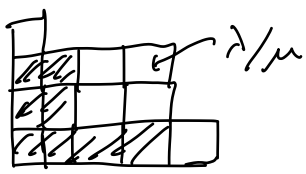
WTS $\langle s_\lambda, e_\mu \rangle = K_{\lambda^T\mu}$

$e_\mu = e_{\mu_1} e_{\mu_2} \cdots e_{\mu_r}$ Need "Pieri rule" to finish.

Pieri Rules: for computing $e_r \cdot s_\lambda$ in Schur basis
and $h_r \cdot s_\lambda$ in Schur basis

Def: A *skew shape* is a diagram of boxes formed by subtracting those of one Young diagram from another. If $\mu \subseteq \lambda$, write λ/μ for "λ skew μ".

Ex: $\lambda = (5,4,4,1)$, $\mu = (4,2,2)$



Def: A skew shape is a *horizontal strip* if no two boxes are in the same column.
A *vertical strip* if no two in the same row.

Thm: We have

$$e_r \cdot s_\lambda = \sum_{\substack{\gamma/\lambda \\ \text{vertical} \\ \text{strip size } r}} s_\gamma$$

$$h_r \cdot s_\lambda = \sum_{\substack{\gamma/\lambda \\ \text{horizontal} \\ \text{strip size } r}} s_\gamma$$

Pf: $h_r s_\lambda = s_{(r)} \cdot s_\lambda = \left(\sum_{R \in SSYT(r)} x^R \right) \left(\sum_{T \in SSYT(\lambda)} x^T \right)$

$$= \sum_{T \in SSYT(\lambda)} \sum_{R \in SSYT(r)} x^{T \leftarrow R}$$

where $T \leftarrow R$ is the RSK insertion of the row R into T from left to right.

Notice that RSK insertion gives a tableau $S := (T \leftarrow R)$ whose shape differs from that of T by a horizontal strip, by Key Lemma. Moreover, given an SSYT S of shape ν where ν/λ is a horizontal strip, unbumping the strip boxes from right to left extracts a row R and leaves an SSYT T of shape λ, reversing the insertion process.

Therefore the summation above is

$$= \sum_{\substack{S \text{ an SSYT} \\ \text{w/ } sh(S)/\lambda \\ \text{horizontal strip}}} x^S$$

$$= \sum_{\substack{\nu : \nu/\lambda \\ \text{horz strip}}} \sum_{S \in SSYT(\nu)} x^S$$

$$= \sum_{\substack{\nu : \nu/\lambda \\ \text{horz strip}}} s_\nu \qquad \text{as desired.}$$

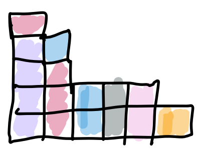
Proof of "e" Pieri rule is similar.

Return to proof that $\omega s_\lambda = s_{\lambda^T}$:

Needed to show $\langle s_\lambda, e_\mu \rangle = K_{\lambda^T \mu}$ = # SSYT's shape λ^T, content μ.

Note $e_\mu = e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k}$.

By Pieri rule, coeff of s_λ in this product is # ways to place vertical strips of sizes $\mu_1, \mu_2, \ldots$ to fill up λ:

 $\leftarrow \mu = (4,4,3,2,2,1)$

Transpose this pic and fill first color w/ 1's, second color w/ 2's, etc.

ways to do this is $K_{\lambda^T \mu}$ = # SSYT's of shape λ^T and content μ. QED.