$\Lambda = \Lambda_{\mathbb{Q}}(x_1, x_2, \ldots) =$ ring of symmetric functions w/ $\mathbb{Q}$ coeffs.

Hall inner product: a <u>bilinear form</u>

$$\langle \, , \, \rangle : \Lambda \times \Lambda \to \mathbb{Q}$$
$$(f, g) \mapsto \langle f, g \rangle$$

Properties of bilinear forms:

- $\langle c_1 f_1 + c_2 f_2, g \rangle = c_1 \langle f_1, g \rangle + c_2 \langle f_2, g \rangle$
- $\langle f, c_1 g_1 + c_2 g_2 \rangle = c_1 \langle f, g_1 \rangle + c_2 \langle f, g_2 \rangle$

$\therefore$ A bilinear form is determined by values of $\langle u_i, v_j \rangle$ where $\{u_i\}$, $\{v_j\}$ are any two bases of the vector space.

<u>Recall</u>: $\{m_\lambda\}$ and $\{h_\lambda\}$ are both bases of Λ.

<u>Def</u>: Hall inner product is the unique bilinear form s.t.

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} := \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases}$$

(m's, h's are called <u>dual</u> <u>bases</u>).

<u>Ex</u>: $\langle s_{(2,1)}, s_{(2,1)} \rangle = \langle m_{(2,1)} + 2m_{(1,1,1)}, \; h_{(2,1)} - h_3 \rangle$

Jacobi-Trudi

$$= \langle m_{(2,1)}, h_{(2,1)} \rangle + 2\langle m_{(1,1,1)}, h_{(2,1)} \rangle - \langle m_{(2,1)}, h_{(3)} \rangle - 2\langle m_{(1,1,1)}, h_3 \rangle$$

$$= 1 + 0 - 0 - 0 = \boxed{1}$$

Prop: $\langle\, ,\rangle$ is symmetric, i.e. $\langle f,g\rangle = \langle g,f\rangle$ for all $f,g\in\Lambda$.

Pf: Suffices to show $\langle h_\lambda, h_\mu\rangle = \langle h_\mu, h_\lambda\rangle$ since $\{h_\lambda\}$ is a basis.

Recall from last semester that

$$h_\lambda = \sum_\mu N_{\lambda\mu} m_\mu$$

where $N_{\lambda\mu}$ = # integer matrices w/ row sums $\lambda_1,\lambda_2,\ldots$ and column sums $\mu_1,\mu_2,\ldots$.

By transposing the matrices, we notice $N_{\lambda\mu} = N_{\mu\lambda}$.

Now:

$$\begin{aligned}
\langle h_\lambda, h_\mu\rangle &= \Big\langle \sum_\nu N_{\lambda\nu} m_\nu, h_\mu \Big\rangle \\
&= \sum_\nu N_{\lambda\nu} \underbrace{\langle m_\nu, h_\mu\rangle}_{0 \text{ if } \nu\neq\mu} \\
&= N_{\lambda\mu} \\
&= N_{\mu\lambda} \\
&= \langle h_\mu, h_\lambda\rangle \quad \text{by same argument.}
\end{aligned}$$

QED

Recall from Wed: $\displaystyle\sum_\lambda m_\lambda(\underline{x})\, h_\lambda(\underline{y}) = \prod_{i,j}\frac{1}{1-x_iy_j} = \Omega.$

Def: $\{u_\lambda\}, \{v_\lambda\}$ are __dual__ __bases__ if $\langle u_\lambda, v_\mu\rangle = \delta_{\lambda\mu}$.

Thm: u, v are dual bases iff

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \Omega.$$

Pf: Write $m_\lambda = \sum_\rho \alpha_{\lambda\rho} u_\rho$ and $h_\mu = \sum_\nu \beta_{\mu\nu} v_\nu$.

Then $\underbrace{\langle m_\lambda, h_\mu \rangle}_{=\,\delta_{\lambda\mu}} = \sum_{\rho,\nu} \alpha_{\lambda\rho} \beta_{\mu\nu} \langle u_\rho, v_\nu \rangle$ $\quad (*)$

Let A = matrix $A_{\rho\nu} = \langle u_\rho, v_\nu \rangle$. Then $(*)$ means

$$I = \alpha A \beta^T \quad \text{where} \quad \left.\begin{array}{l} \alpha = (\alpha_{\lambda\rho}) \\ \beta = (\beta_{\mu\nu}) \end{array}\right\} \text{matrices}$$

So u, v dual $\iff A = I$

$\iff I = \alpha \beta^T$

$\iff I = \alpha^T \beta$ (transpose both sides)

$\iff \boxed{\delta_{\rho\nu} = \sum \alpha_{\lambda\rho} \beta_{\lambda\nu}}$.

Now $\Omega = \sum_\lambda m_\lambda(x) h_\lambda(y) = \sum_\lambda \left(\sum_\rho \alpha_{\lambda\rho} u_\rho(x)\right)\left(\sum_\nu \beta_{\lambda\nu} v_\nu(y)\right)$

$$= \sum_{\rho,\nu} \left(\sum_\lambda \alpha_{\lambda\rho} \beta_{\lambda\nu}\right) u_\rho(x) v_\nu(y)$$

if dual $\rightarrow$ $= \sum_{\rho,\nu} \delta_{\rho\nu} u_\rho(\underline{x}) v_\nu(y)$

$$= \sum_\nu u_\nu(x) v_\nu(y) \quad ✓ \ \square$$

Cor: $\langle \frac{p_\lambda}{z_\lambda}, p_\mu \rangle = \delta_{\lambda\mu}$ (recall $z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots$, $m_i = \#$ i's in λ)

Pf: $\{\frac{p_\lambda}{z_\lambda}\}$ still a basis, just renormalized.

On Wednesday, proved $\sum \frac{p_\lambda(x)}{z_\lambda} p_\lambda(y) = \Omega$.

Use above Thm. $\square$

Cor: $\langle \, , \rangle$ is positive definite, i.e. $\langle f, f \rangle \geq 0$ for all f, equality iff $f=0$

Pf: Write $f = \sum c_\lambda p_\lambda$ in p basis.

Then $\langle f, f \rangle = \sum c_\lambda^2 z_\lambda$ (since $\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda\mu}$)

≥ 0

Since each z_λ nonzero, $\langle f, f \rangle = 0$ iff all c_λ's are 0, i.e. if $f = 0$.

Cor: Involution ω is an isometry:

$$\langle \omega f, \omega g \rangle = \langle f, g \rangle$$

Pf: Suffices to check $\langle \omega p_\lambda, \omega p_\mu \rangle$ by linearity of ω and $\langle \, , \rangle$. From last class: $\omega p_\lambda = (-1)^{n-k_\lambda} p_\lambda$ where $k_\lambda = \#$ parts of λ. So

$$\langle \omega p_\lambda, \omega p_\mu \rangle = \langle (-1)^{n-k_\lambda} p_\lambda , (-1)^{n-k_\mu} p_\mu \rangle = (-1)^{2n-k_\lambda-k_\mu} \delta_{\lambda\mu}$$

$$= \begin{cases} 0 & \text{if} \quad \lambda \neq \mu \\ 1 & \text{if} \quad \lambda = \mu \end{cases}$$

$$= \delta_{\lambda\mu} \,.$$

QED