Math 502: Combinatorics
Homework 9

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1+) [2 points] Prove that the tensor product of two Hadamard matrices is a Hadamard matrix.
2. (2-) [3 points] Prove that there is only one $2-(7,3,1)$-design up to isomorphism.
3. (2-) [3 points] Enumerate the automorphisms of the Fano plane.
4. (1+) [2 points] The **complementary design** to a design $\mathcal{D}=(X,\mathcal{B})$ is the pair $\mathcal{D}^c=(X,\mathcal{B}^c)$ where the $\mathcal{B}^c=\{X\setminus B : B\in\mathcal{B}\}$. Show that if $\mathcal{D}$ is a $1-(v,k,\lambda)$ design then $\mathcal{D}^c$ is a $1-(v,v-k,v\lambda/k-\lambda)$ design.
5. (2-) [3 points] Prove the converse of the connection between Hadamard matrices and designs that we discussed in class. That is, prove that any $2-(4t-1,2t-1,t-1)$ design $\mathcal{D}$ gives rise to a Hadamard matrix H whose associated design is $\mathcal{D}$. Also prove that for any such design, there is a corresponding $2-(4t-1,2t,t)$ design (see previous problem).
6. (2-) [3 points] Prove that the edge-complement of a strongly regular graph is strongly regular, and find the new parameters in terms of the previous.
7. (3-) [8 points] Prove that there is a unique $2-(11,5,2)$ design up to isomorphism. Then, let X be the set of all points AND blocks of the $2-(11,5,2)$ design. Let $\mathcal{B}$ consist of the 6-element subsets of X of the following three types:
   - A point and the five blocks containing it;
   - A block and the five points contained in it;
   - An *oval* and its *tangents* (an *n-arc* is a set of n points, no three of which are contained in a block, a *tangent* of an n-arc S is a block B such that $|S\cap B|=1$. An *oval* is an n-arc that either has no tangents, or each point of the arc lies on a unique tangent.)

   Prove that (X,B) is a $3-(22,6,1)$ design.