Math 502: Combinatorics
Homework 5

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1+) [2 points] Draw the full crystal of all semistandard skew tableaux of skew shape $(3,2)/(1)$ using letters $1,2,3$ to fill the boxes. (Note that some of these components are not vertically symmetric; this is because we're using 4 boxes but only 3 letters.) Deduce the Schur decomposition of $s_{(3,2)/(1)}(x_1,x_2,x_3)$.

2. (1+) [2 points] Draw the full crystal of words of length 4 that only use the letters 1 and 2 (using only the E_1 and F_1 operators).

3. (2-) [3 points] Prove that a word w is highest weight (i.e., $E_i(w)=0$ for all i) if and only if w is reverse ballot.

4. (2) [3 points] Formulate and prove a ballot-type condition for w to be *lowest weight*, that is, $F_i(w)=0$ for all $i=1,2,\ldots,n$. Such a word is called *anti-ballot*.

5. (1+) [2 points] Show that the only highest weight tableau on a straight shape λ is the one in which row i is filled with all i's. Use this to calculate the Littlewood-Richardson coefficient

$$c^{\lambda}_{\emptyset,\nu}$$

in terms of λ and ν.

6. (1+) [2 points] Show that for any symmetric function f, we have

$$\langle s_\mu \cdot f, s_\lambda\rangle = \langle f, s_{\lambda/\mu}\rangle.$$

7. (2+) [4 points] How many ballot words of length n have only 1's and 2's?

8. (3-) [8 points] How many ballot words of length n have only the letters $1,2,3$?

9. (5) [∞ points] When are two skew Schur functions equal? In other words, determine combinatorial conditions on partitions λ,μ,ρ,θ that determine precisely when

$$s_{\lambda/\mu} = s_{\rho/\theta}.$$