Math 502: Combinatorics
Homework 14

This homework is a review homework, and serves as a mini-final exam for the course. Every problem is worth 1 point and therefore you must do all of them to earn the full 10 points!

Problems

1. Show that, if we are working in the symmetric function ring $\Lambda_{\mathbb{F}_q}(x_1, x_2, x_3, \ldots)$ over the finite field $\mathbb{F}_q$, we have the equality of power sum symmetric functions $p_q = p_{(1^q)}$ where (1^q) denotes the partition $(1, 1, 1, \ldots, 1)$ of length q. Conclude that p_λ is not a basis of symmetric functions in characteristic p.

2. Compute $\langle h_{(3,2,2)}, s_{(2,2,1,1,1)} \rangle$ where $\langle , \rangle$ is the Hall inner product. (You may not use a computer for this problem, you must do it by hand and show your work - though you may use a computer to check your answer.)

3. The **charge** of a permutation $w \in S_n$ is defined as follows. Label the entry 1 of w with a subscript 0. Then, for each entry $i = 2, 3, 4, \ldots, n$, if i is to the right of $i - 1$ in w then label it with one more than the subscript that labeled $i - 1$, and otherwise label it with the same subscript. Then the charge is the sum of the subscript labels.

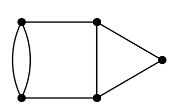
   Prove that charge is invariant under Knuth equivalence and therefore can be viewed as a statistic on the RSK insertion tableau of the permutation.

4. Show that
$$\sum_{w \in S_n} q^{\mathrm{charge}(w)} = (n)_q!$$
   where charge is as defined in the previous problem.

5. Let V be the vector space over $\mathbb{C}$ whose basis consists of all binary strings of length 4 (so an element of V is a formal linear combination of binary strings of length 4). Consider the action of the symmetric group S_4 on V by permuting the entries of any given binary string according to the permutation. Decompose V into irreducible S_4 representations. (Hint: You may want to find the character of V first, and use the character table of S_4. There also may be an initial partial decomposition you can do to simplify your work.)

6. Prove that the set of k-flats in $\mathbb{P}^n_{\mathbb{F}_q}$ form a $2 - (v, k, \lambda)$ design for some v, k, λ, and find the parameters v, k, λ in terms of n, k, q.

7. Compute the number of bases in the graphical matroid associated to the following graph (Hint: You may want to use a result discussed in 501, and you may use a computer):



8. We drew $\mathbb{P}^2_{\mathbb{F}_2}$ as the Fano plane. Draw $\mathbb{P}^2_{\mathbb{F}_3}$ in a similar manner, by drawing all of its points and all of its lines.

9. Show $\mathrm{Gr}(n, k) \cong \mathrm{Gr}(n, n-k)$ over any field.

10. Compute the product of Schubert classes $\sigma^3_{(2,1)}$ in the Chow ring $A^*(\mathrm{Gr}(3, 6))$.