Math 502: Combinatorics
Homework 13

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (2-) [3 points] The **Plücker embedding** of the Grassmannian is the map $\mathrm{Gr}(k, n) \to \mathbb{P}^{\binom{n}{k}-1}$ whose homogeneous coordinates are given by the $k \times k$ minor (determinants) of any matrix representing an element of $\mathrm{Gr}(k, n)$.

   Prove that the map is well-defined independent of the representing matrix.

2. (1+) [2 points] Prove that any two lines in $\mathbb{P}^2$ are rationally equivalent.

3. (2+) [4 points] Prove that the class of a line generates $A^1(\mathbb{P}^2)$.

4. (1+) [2 points] For $\mathrm{Gr}(2, 4)$, suppose x_{ij} is the Plücker coordinate corresponding to the 2×2 determinant of the i-th and j-th columns. Prove that

$$x_{12}x_{34} + x_{14}x_{23} = x_{13}x_{24}.$$

   (This is called a **Plücker relation**. In general there is a set of quadratic Plücker relations that define the Grassmannian as a projective variety.)

5. (2-) [3 points] Prove, using a dimension analysis, that the single Plücker relation above is all that is needed to define the Grassmannian $\mathrm{Gr}(2, 4)$ as a subvariety of $\mathbb{P}^5$.

6. (1+) [2 points] (Warmup for the next two problems.) Define the *Bruhat order* to be the partial order on S_n defined as follows. Write each permutation as a *reduced word*: a product of transpositions $s_i = (i\ i + 1)$ of minimal possible length. Then $\pi \le \sigma$ if there is a reduced word for π that is a (not necessarily consecutive) substring of a reduced word for σ. Draw the Bruhat order poset for S_3.

7. (2) [3 points] Draw the Bruhat order poset for S_4. (See previous problem).

8. (3-) [8 points] Prove that the closure of a Schubert cell Ω_π° in the flag variety is the disjoint union of all Schubert cells corresponding to the permutations above π in the Bruhat order defined in the previous problem.

9. (5) [∞ points] The Chow ring (or cohomology ring) of the flag variety can be shown to be the *coinvariant ring*

$$R_n = \mathbb{C}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$$

   where $e_1, \ldots, e_n$ are the elementary symmetric functions in the x variables. The polynomials corresponding to the Schubert varieties X_π are the *Schubert polynomials* S_π. Find a combinatorial rule for the product of Schubert polynomials in terms of Schuberts analogous to the Littlewood-Richardson rule:

$$S_\pi S_\sigma = \sum_\rho g^\rho_{\pi\sigma} S_\rho$$