Math 502: Combinatorics
Homework 11

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (1) [1 point] Prove that if M is a matroid, the dual matroid M^* defined in class (formed by replacing the bases with their complements) is a matroid.

2. (1+) [2 points] Draw the lattice of flats of the graphical matroid whose graph is K_4.

3. (2) [3 points] Prove that, in the lattice of flats of a matroid, if X and Y are flats, then $X \vee Y = \mathrm{cl}(X \cup Y)$ and $X \wedge Y = X \cap Y$. You may use any of the matroid axioms.

4. (1+) [2 points] Prove that representable matroids satisfy the rank semimodular inequality for the lattice of flats: if $E = \{v_1, \ldots, v_n\}$ is a set of vectors in $\mathbb{R}^m$, then if X, Y are flats in E, we have $\mathrm{rk}(X \cap Y) + \mathrm{rk}(\mathrm{cl}(X \cup Y)) \leq \mathrm{rk}(X) + \mathrm{rk}(Y)$.

5. (1) [1 point] Suppose $\{x, y\}$ and $\{y, z\}$ are both circuits of a matroid and none of x, y, z are loops (circuits by themselves). Show that $\{x, z\}$ is a circuit. (You may use the circuit axioms.)

6. (2-) [3 points] Given a matroid $M = (E, \mathcal{B})$ defined by the basis axioms, define the associated simple matroid $\overline{M}$ to be formed from M by:

   - Removing all loops from E
   - Removing one element of each 2-circuit from E arbitrarily (i.e. choose one element from each *parallel class* - equivalence class of elements in 2-circuits with each other - to keep)
   - Defining $\overline{B}$ to be the set of bases formed by replacing any element of a parallel class that was in a basis B by the corresponding element from that parallel class

   If the remaining elements form a set $\overline{E}$, define $\overline{M} = (\overline{E}, \overline{B})$. Show that $\overline{M}$ is a simple matroid. (You may use the previous problem statement even if you did not do it.)

7. (2) [3 points] Let $(E, \mathcal{I})$ be a matroid and let $w : E \to \mathbb{R}$ be any weight function. Show that applying the greedy algorithm to find a basis of minimal weight does indeed find a basis of minimal weight.

8. (3) [9 points] Look up the *Vámos matroid*. Prove that it is not representable over any field.