Math 502: Combinatorics
Homework 1

Recall that you must hand in a subset of the problems for which deleting any problem makes the total score less than 10. The maximum possible score on this homework is 10 points. See the syllabus for scoring details.

Problems

1. (2) [3 points] Show that every polynomial in two variables x, y can be written uniquely as a sum of a (two variable) symmetric polynomial and a (two variable) antisymmetric polynomial.

2. (1+) [2 points] (Review.) Write the power sum symmetric function p_3 in terms of elementary symmetric functions.

3. (1+) [2 points] Use the ω involution discussed in class (to be discussed on Wed. Jan 25) to use the formula you obtained in the previous problem to express p_3 in terms of homogenous symmetric functions.

4. (1+) [2 points] Determine whether each of $e_{(3,2,2)}, h_{(3,2,2)}, p_{(3,2,2)}$, and $m_{(3,2,2)}$ are Schur positive. (You can use Sage and the symmetric functions package.)

5. (1+) [2 points] Compute the Schur polynomial $s_{(2,1)}(x_1, x_2, x_3)$ as a ratio of determinants (i.e. a ratio of antisymmetric functions, to be discussed in class on Monday Jan 23), and compute it using semistandard Young tableaux, and show that the two computations agree.

6. (2-) [3 points] Prove that

$$\sum_{\lambda} m_{\lambda}(x_1, x_2, \ldots) e_{\lambda}(y_1, y_2, \ldots) = \prod_{i,j=1}^{\infty} (1 + x_i y_j).$$

7. (2+) [4 points] Stanley chapter 7 problem 7.3. (Scroll down to next two pages of this document for photos of the problems from Stanley)

8. (2+) [4 points] Stanley chapter 7 problem 7.4.

9. (2+) [4 points] Stanley chapter 7 problem 7.5.

10. (2+) [4 points] Stanley chapter 7 problem 7.6.

11. (2+) [4 points] Stanley chapter 7 problem 7.7.

12. (2+) [4 points] Stanley chapter 7 problem 7.8.

13. (2+) [4 points] Stanley chapter 7 problem 7.9.

7.2. Let Par(n) denote the set of all partitions of n with the dominance ordering.

a. [2] Show that Par(n) is a lattice.

b. [2+] Show that Par(n) is self-dual.

c. [2+] Find the smallest value of n for which Par(n) is not graded.

d. [2+] Show that the maximum number of elements covered by an element of Par(n) is $\lfloor \frac{1}{2}(\sqrt{1+8n}-3) \rfloor$.

e. [2+] Show that the shortest maximal chain in Par(n) has length $2n-4$ for $n \geq 3$.

f. [3–] Show that the longest maximal chain in Par(n) has length

$$\tfrac{1}{3}m(m^2+3r-1) \sim \tfrac{1}{3}(2n)^{3/2},$$

where $n = \binom{m+1}{2} + r$, $0 \leq r \leq m$.

7.3. [2+] Expand the power series $\prod_{i\geq 1}(1+x_i+x_i^2)$ in terms of the elementary symmetric functions.

7.4. [2+] Show that

$$h_r(x_1,\ldots,x_n) = \sum_{k=1}^{n} x_k^{n-1+r} \prod_{i\neq k} (x_k - x_i)^{-1}.$$

7.5. [2+] Prove the identity

$$\left(1 - \sum_{n\geq 1} p_n t^n\right)^{-1} = \frac{\sum_{n\geq 0} h_n t^n}{1 - \sum_{n\geq 1}(n-1)h_n t^n}. \tag{7.165}$$

7.6. [2+] Let $w \in \mathfrak{S}_n$ have cycle type λ. Give a direct bijective proof of Proposition 7.7.3, i.e., the number of elements $v \in \mathfrak{S}_n$ commuting with w is equal to $z_\lambda = 1^{m_1} m_1! \, 2^{m_2} m_2! \cdots$, where $m_i = m_i(\lambda)$.

7.7. [2+] Let Ω^n denote the subspace of Λ^n consisting of all $f \in \Lambda^n$ satisfying

$$f(x_1, -x_1, x_3, x_4, \ldots) = f(x_3, x_4, \ldots).$$

For instance, $m_1 = x_1 + x_2 + \cdots \in \Omega^1$. Find a "simple" basis for Ω^n. Express the dimension of Ω^n in terms of the number of partitions of n with a suitable restriction.

7.8. [2+] Let $f \in \Lambda^n$, and for any $g \in \Lambda^n$ define $g_k \in \Lambda^{nk}$ by

$$g_k(x_1, x_2, \ldots) = g\left(x_1^k, x_2^k, \ldots\right).$$

Show that

$$\omega f_k = (-1)^{n(k-1)}(\omega f)_k.$$

7.9. [2+] Let λ be a partition of n of length ℓ. Define the *forgotten symmetric function* f_λ by

$$f_\lambda = \varepsilon_\lambda\, \omega(m_\lambda),$$

where $\varepsilon_\lambda = (-1)^{n-\ell}$ as usual. (Sometimes f_λ is defined just as $\omega(m_\lambda)$.) Let $f_\lambda = \sum_\mu a_{\lambda\mu} m_\mu$. Show that $a_{\lambda\mu}$ is equal to the number of distinct permutations $(\alpha_1, \alpha_2, \ldots, \alpha_\ell)$ of the sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ such that

$$\{\alpha_1 + \alpha_2 + \cdots + \alpha_i : 1 \le i \le \ell\} \supseteq \{\mu_1 + \mu_2 + \cdots + \mu_j : 1 \le j \le \ell(\mu)\}.$$

For instance, if $\lambda = (3, 2, 1, 1)$ and $\mu = (5, 2)$, then $a_{\lambda\mu} = 5$, corresponding to (3, 2, 1, 1), (2, 3, 1, 1), (1, 1, 3, 2), (1, 3, 1, 2), and (3, 1, 1, 2).

7.10. [3−] Let $\lambda \in \mathrm{Par}$, and define the symmetric power series

$$A_\lambda(x) = \prod_\alpha \left(1 - x^\alpha\right)^{-1}$$

$$B_\lambda(x) = \prod_\alpha \left(1 + x^\alpha\right),$$

where α ranges over all *distinct* permutations of $(\lambda_1, \lambda_2, \ldots)$. Find a formula for $\omega A_\lambda(x)$ and $\omega B_\lambda(x)$ in terms of $A_\mu(x)$'s and $B_\mu(x)$'s. For instance,

$$\omega A_1(x) = B_1(x)$$
$$\omega A_{11}(x) = A_2(x) A_{11}(x)$$
$$\omega A_2(x) = A_2(x)^{-1}.$$

In general, express the answer in terms of the coefficients $a_{\lambda\mu}$ defined in Exercise 7.9.

7.11. [2+] Let q be an indeterminate. Find the Schur function expansion of $\sum_{\mu \vdash n}$ $q^{\ell(\mu)-1} m$ ... ...ition 7.10.5, i.e., if $\mu, \lambda \vdash n$ and $\mu \le \lambda$