RSK and increasing/decreasing subsequences:

Def: A longest increasing subsequence of a word $w \in (\mathbb{Z}_+)^n$ is a subsequence $w_{i_1} \le w_{i_2} \le \cdots \le w_{i_\ell}$ with $i_1 < i_2 < \ldots < i_\ell$ with ℓ as large as possible.

A longest decreasing subsequence is
$w_{i_1} > w_{i_2} > \cdots > w_{i_d}$ w/ $i_1 < i_2 < \ldots < i_d$
with d as large as possible.

Write $\ell(w) = \ell$ and $d(w) = d$.

Note: Both def's above are strictly increasing/decreasing with respect to standardization order.

> **Thm:** Let w be a word and $S = \mathrm{ins}(w)$ the RSK insertion tableau of w, $sh(S) = \lambda$. Then $\ell(w) = \lambda_1$ and $d(w) = \lambda^t_1$.

Ex: $8\,2\,3\,5\,7\,1\,4\,6 \xrightarrow{RSK} \left(3\left\{ \underbrace{\begin{array}{llll} 8 & & & \\ 2 & 5 & 7 & \\ 1 & 3 & 4 & 6 \end{array}}_{4} \right. , \begin{array}{llll} 6 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 5 \end{array} \right)$

$\ell = 4 \quad d = 3$

Ex (destandardized)

$6\,2\,2\,3\,5\,1\,2\,4 \xrightarrow{RSK} \left(\begin{array}{llll} 6 & & & \\ 2 & 3 & 5 & \\ 1 & 2 & 2 & 4 \end{array} , \begin{array}{llll} 6 & & & \\ 2 & 7 & 8 & \\ 1 & 3 & 4 & 5 \end{array} \right)$

Note: Due to standardization map commuting w/ RSK, suffices to prove thm for permutations.

Lemma: Reading word of T inserts to T (w/ a specific recording tableau)

Pf/Ex: $\underbrace{8}\,\underbrace{2\,5\,7}\,\underbrace{1\,3\,4\,6} \xrightarrow{RSK} \left(\begin{array}{llll} 8 & & & \\ 2 & 5 & 7 & \\ 1 & 3 & 4 & 6 \end{array} \; , \; \begin{array}{llll} \boxed{5} & & & \\ \boxed{2} & \boxed{6} & \boxed{7} & \\ \boxed{1} & \boxed{3} & \boxed{4} & \boxed{8} \end{array} \right)$

$\parallel$

$rw\left(\begin{array}{llll} 8 & & & \\ 2 & 5 & 7 & \\ 1 & 3 & 4 & 6 \end{array} \right)$

Layer horizontal strips of sizes $\lambda_d, \lambda_{d-1}, \dots \lambda_1$ in order

□

Lemma: Length ℓ of longest increasing subsequence of $rw(T)$ is λ_1 where $\lambda = sh(T)$.

reading word

Similarly $d(rw(T)) = \lambda^T_1$.

Pf: Starting w/ any entry of T, to find an entry greater than it that is later in reading order, need to move to next column to the right (by standard condition). So length ℓ is at most the number of columns of T. Bottom row works, so $\ell = \lambda_1$.

Similarly, need to move downwards for decreasing subsequence, left column works $\Rightarrow d = \lambda^T_1$.

□

Now: When do two permutations have the same insertion tableau T? (possibly different recording tableaux)

Ex:

$$132 \to \left(\begin{array}{ll} 3 & \\ 1 & 2 \end{array}, \begin{array}{ll} 3 & \\ 1 & 2 \end{array}\right)$$

$$312 \to \left(\begin{array}{ll} 3 & \\ 1 & 2 \end{array}, \begin{array}{ll} 2 & \\ 1 & 3 \end{array}\right)$$

$$213 \to \left(\begin{array}{ll} 2 & \\ 1 & 3 \end{array}, \begin{array}{ll} 2 & \\ 1 & 3 \end{array}\right)$$

$$231 \to \left(\begin{array}{ll} 2 & \\ 1 & 3 \end{array}, \begin{array}{ll} 3 & \\ 1 & 2 \end{array}\right)$$

Two equivalent pairs above are foundational; all other equivalent words are similar.

Knuth equivalence

Def: A <u>Knuth move</u> on a permutation swaps two adjacent letters according to the following rule: If $a<b<c$ and one of:

acb, cab, bac, or bca

appears consecutively, then we can swap a and c.

<u>In other words</u>: Can swap a, c if either the entry to the right of the pair or to the left of the pair is between a and c in magnitude.

Def: π and w are <u>Knuth equivalent</u> if they can be obtained from one another by a sequence of Knuth moves.

Ex: Knuth equivalence classes of size 4 and their insertion tableaux:

$\{1234\} \xrightarrow{\text{ins}} \boxed{1234}$ $\qquad$ $\{4321\} \xrightarrow{\text{ins}} \begin{array}{|c|}\hline 4\\ 3\\ 2\\ 1\\ \hline\end{array}$

its own Knuth equivalence class

$$\{1243,\ 1423,\ 4123\} \xrightarrow{\text{ins}} \begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array}$$

$$\{1324,\ 1342,\ 3124\} \xrightarrow{\text{ins}} \begin{array}{lll} 3 & & \\ 1 & 2 & 4 \end{array}$$

$$\{1423,\ 4123,\ 1243\} \xrightarrow{\text{ins}} \begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array}$$

$$\{3412,\ 3142\} \xrightarrow{\text{ins}} \begin{array}{ll} 3 & 4 \\ 1 & 2 \end{array}$$

$$\{2413,\ 2143\} \xrightarrow{\text{ins}} \begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}$$

$$\vdots$$

Thm: Two permutations π and w have the same insertion tableau iff they are Knuth equivalent.

Proof: First we show a Knuth move on the last three letters does not change the insertion tableau of a permutation; by induction on the length of the word, this suffices.

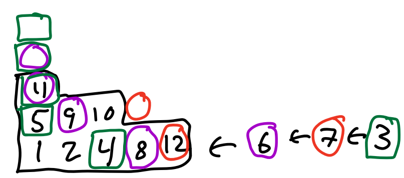
Suppose $\pi = \pi_1 \cdots \pi_n$ and $\pi_{n-2}\pi_{n-1}\pi_n = bca$ with $a<b<c$.

Starting with $T' = \mathrm{ins}(\pi') = \mathrm{ins}(\pi_1 \cdots \pi_{n-3})$, we wish to show that

$$T' \leftarrow \boxed{b} \leftarrow \boxed{c} \leftarrow \boxed{a}$$

$$= T' \leftarrow \boxed{b} \leftarrow \boxed{a} \leftarrow \boxed{c}.$$

Ex:



In the first insertion, the insertion path of b is strictly left of that of c, by Key lemma 2.

Then, insertion path of a is weakly left of that of b since $a<b$ and c's, b's paths don't collide (see exercises on inserting $b \geq a$ in order on Hwk 3). Thus the insertion paths of a, c are disjoint, and so we can switch the order in

which we insert them and end up with the same tableau.

Now consider

$$T' \leftarrow \boxed{c} \leftarrow \boxed{a} \leftarrow \boxed{b}$$

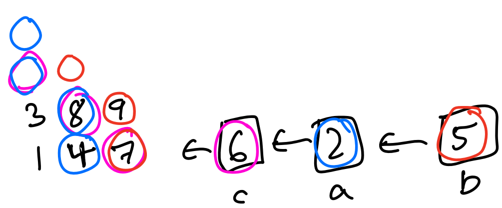
and

$$T' \leftarrow \boxed{a} \leftarrow \boxed{c} \leftarrow \boxed{b} .$$

In the first, the insertion path of c is weakly left of that of a; if it is strictly left, we can switch as before.

Otherwise, the insertion paths intersect in some earliest row r in box $\boxed{x}$, where the c path bumps an entry c_r into box x, and then the a path bumps an entry a_r into box x.

Then $a_r < c_r$, and since $b > a$ the 'b' path lies strictly right of the 'a' path, so the r^{th} entry bumped in the "b" path, b_r (if it exists) is greater than a_r. But since $b < c$, the b path below row r is also weakly to the left of the c path; at row r, since a_r bumps c_r, the b_r entry bumps the box $\boxed{y}$ just to the right of $\boxed{x}$ (or lands there if it is empty).



$\boxed{x} = 8$ $\boxed{y} = 9$

Now, if we switch the order of inserting a and c, the rows below r will be unchanged, and for row r, we first have a_r bumping the entry in box $\boxed{x}$, then c_r bumps $\boxed{y}$ (or lands there if it is an empty box), and then in the b path, b_r bumps c_r out of box $\boxed{y}$. The result is the same new row r, and this then continues in higher rows.

Now we show the reverse direction: that two words w/ the same insertion tableau are Knuth equivalent. It suffices to show that if $ins(w) = T$, then $w \cong rw(T)$ by the reading word lemmas. We do so by induction on the length of w.

<u>Base case:</u> $w = 1 \overset{ins}{\rightsquigarrow} \boxed{1}$
$\cong rw(\boxed{1})$

<u>Ind. step:</u> Assume $\left(ins(w') = T' \Rightarrow w' \cong rw(T')\right)$ for all words w' of length $\leq n-1$.

Let w be a permutation of length n, with last entry $w_n = b$.

Let $T' = ins(w_1 \cdots w_{n-1})$; by induction,

$$w_1 \cdots w_{n-1} \cong rw(T') =: \underbrace{r_1^{(k)} \cdots r_{\lambda_k}^{(k)}}_{\text{top row}} \underbrace{r_1^{(k-1)} \cdots r_{\lambda_{k-1}}^{(k-1)}}_{\text{second row}} \cdots \underbrace{r_1^{(1)} \cdots r_{\lambda_1}^{(1)}}_{\text{bottom row}}$$

We wish to show that $\left(r_1^{(k)} \cdots r_{\lambda_1}^{(1)}\, b\right)$, which is Knuth equivalent to w, is Knuth equivalent to $rw(T) = rw(ins(w)) = rw(T' \leftarrow \boxed{b})$.

Inserting b into T', we first bump out some $c = r^{(1)}_{i_1}$ from the bottom row; we can do a sequence of Knuth moves

$$r_1^{(b)} \cdots \underbrace{r_1^{(1)} r_2^{(1)} \cdots c\, r^{(1)}_{i_1+1} \cdots \overbrace{r^{(1)}_{\lambda_1-1} r^{(1)}_{\lambda_1} b}^{\text{Knuth move}}}_{\text{bot row}}$$

$$\sim r_1^{(b)} \cdots r_1^{(1)} r_2^{(1)} \cdots c\, r^{(1)}_{i_1+1} \cdots r^{(1)}_{\lambda_1-1}\, b\, r^{(1)}_{\lambda_1}$$

$$\sim r_1^{(b)} \cdots r_1^{(1)} r_2^{(1)} \cdots c\, r^{(1)}_{i_1+1} \cdots b\, r^{(1)}_{\lambda_1-1} r^{(1)}_{\lambda_1}$$

$$\vdots$$

$$\sim r_1^{(b)} \cdots r_1^{(1)} r_2^{(1)} \cdots c\, b\, r^{(1)}_{i+1} \cdots$$

And now we can switch c past entries to its left using w_n to start, until we switch c past everything up to the entry d that it bumps in the next row, and so on. $\square$

Ex:

$$\begin{array}{llll} 6 \\ 4 & 7 \\ 1 & 2 & 5 & 8 \end{array} \leftarrow \boxed{3} \qquad rw \leftarrow 3 = 6\,4\,\underset{d}{\underline{7}}\,1\,2\,\underset{c}{\underline{5}}\,8\,\underset{b}{\underline{3}}$$

$$\overset{211}{\wr\wr}$$

$$6\,4\,\underset{d}{\underline{7}}\,1\,2\,\underset{c}{\underline{5}}\,\underset{b}{\underline{3}}\,8$$

$$\overset{211}{\wr\wr}$$

$$6\,4\,\underset{d}{\underline{7}}\,1\,\underset{c}{\underline{5}}\,2\,\underset{b}{\underline{3}}\,8$$

$$\begin{array}{l} 6\ 7 \\ 4\ 5 \\ 1\ 2\ 3\ 8 \end{array} \xrightarrow{\ rw\ } 6\,\underset{d}{\underline{7}}\,4\,\underset{c}{\underline{5}}\,1\,2\,\underset{b}{\underline{3}}\,8 = rw\left(\begin{array}{l} 6 \\ 4\ 7 \\ 1\ 2\ 5\ 8 \leftarrow 3 \end{array}\right)$$

$$\|$$

$$6\,4\,\underset{\cancel{d}}{\underline{7}}\,\underset{c}{\underline{5}}\,1\,2\,\underset{b}{\underline{3}}\,8$$

$$7\,\|$$

<u>Lemma:</u> Longest increasing subseq. length is invariant under Knuth equiv.

<u>Pf:</u> Given an increasing subsequence, if a Knuth move changes two of its entries $a < c$:

<u>Case 1:</u> b to the right of a, c: Replace c in the subsequence with b. Still an increasing subseq of same length in $\ldots c\underline{a}\underline{b} \ldots$ as in $\ldots \underline{a}c\underline{b} \ldots$

<u>Case 2:</u> b to the left of a, c:

$\ldots b\,\underline{a}\,\underline{c} \ldots$

$\to \ldots \underline{b}\,\underline{c}\,a \ldots$

Replace a in the subsequence with b. Still valid and same length.

$\square$

Now, pf of Longest Inc Subseq. thm:

Let $T = \text{ins}\ w$, $\lambda = sh(T)$

$$\left.\begin{array}{l} \text{Then } w \cong rw(T). \\ \text{So } \ell(w) = \ell(rw(T)). \\ \qquad\quad = \lambda_1 \end{array}\right\} \text{by previous lemmas}$$

QED

Proof of decreasing subsequences similar.

Mega-Theorem (generalization). A longest i-chain of increasing subsequences of w consists of:

- An increasing subseq s_1 of w
- An " " s_2 of $w \setminus s_1$

  ⋮

- An inc. subseq s_i of $w \setminus (s_1 \cup s_2 \cup \cdots \cup s_{i-1})$

w/ maximal total length $l_i = |s_1| + |s_2| + \cdots + |s_i|$.

The length of the longest i-chain l_i is given by

$$\lambda_1 + \lambda_2 + \cdots + \lambda_i.$$

(Pf omitted)

Jeu de Taquin ("The teasing game")

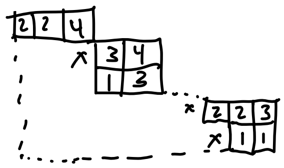
Motivation: To show $\emptyset \xleftarrow{ins} w \xleftarrow{ins} (\rho \xleftarrow{ins} \pi)$

$= \emptyset \xleftarrow{ins} (w \xleftarrow{ins} \rho) \xleftarrow{ins} \pi$

(Associative operation; forms the plactic monoid)

Def: A skew semistandard Young tableau is a filling of the boxes of a skew shape λ/μ w/ positive integers s.t. rows weakly increase $\rightarrow$

cols strictly increase $\uparrow$

Ex:  $\leftarrow$ skew shape $(9,9,5,5,3)/(7,6,3,3)$

x = inner corner

Def: A corner of a Young diagram μ is a square at the top of its column and right end of its row.

An inner corner of λ/μ is a corner of μ.

An outer corner of λ/μ is a square outside of λ just above a column of λ and just right of a row.

Def: An (inner) jeu de taquin slide into an inner corner $\boxed{x}$ of a skew SSYT T consists of the following process:

1. Compare the square $\boxed{b}$ to the right of $\boxed{x}$ to the one above $\boxed{a}$. Slide $\boxed{a}$ down if $a \le b$ or if $\boxed{b}$ does not exist; otherwise slide $\boxed{b}$ left into the empty square $\boxed{x}$.
   $\begin{array}{|c|c|} \hline a & \\ \hline x & b \\ \hline \end{array}$
2. Whatever square was vacated by $\boxed{a}$ or $\boxed{b}$, label this new empty square by $\boxed{x}$.
3. Repeat steps 1 and 2 until the square $\boxed{x}$ has empty squares to its right and above.

Ex:
$$\begin{array}{ccccc} 2 & 4 & 4 & 5 & \\ & 3 & 3 & 3 & 6 \\ & 1 & 2 & 2 & 3 \\ (x) & 1 & 1 & 2 & \end{array} \longrightarrow \begin{array}{ccccc} 2 & 4 & 4 & 5 & \\ & 3 & 3 & 3 & 6 \\ \boxed{x} & 2 & 2 & 3 & \\ 1 & 1 & 1 & 2 & \end{array} \longrightarrow \begin{array}{ccccc} 2 & 4 & 4 & 5 & \\ & 3 & 3 & 3 & 6 \\ 2 & \boxed{x} & 2 & 3 & \\ 1 & 1 & 1 & 2 & \end{array}$$

$$\to \begin{array}{ccccc} 2&4&4&5& \\ &3&3&3&6 \\ &2&2&\boxed{x}&3 \\ &1&1&1&2 \end{array} \longrightarrow \begin{array}{ccccc} 2&4&4&5& \\ &3&3&\boxed{x}&6 \\ &2&2&3&3 \\ &1&1&1&2 \end{array} \longrightarrow \begin{array}{ccccc} 2&4&4&(x)& \\ &3&3&5&6 \\ &2&2&3&3 \\ &1&1&1&2 \end{array}$$

An <u>outer</u> JDT slide reverses these steps, starting with an outer corner.

<u>Observation:</u> JDT slides send skew SSYT's to skew SSYT's.

<u>Def:</u> The <u>reading word</u> of any labeled diagram of grid squares (even skew shapes or partial steps of JDT) is formed by reading the rows from top to bottom, and left to right within each row.

<u>Lemma:</u> If T and S are skew tableaux w/ S obtained from T by a sequence of JDT slides, then $rw(T)$ is Knuth equivalent to $rw(S)$.

<u>Pf:</u> When the $\boxed{x}$ slides horizontally, the reading word is unchanged.

When the $\boxed{x}$ slides vertically: it looks like

$$\begin{array}{cccccccc} 3&3&4&4&4&6&& \\ &&3&3&\boxed{x}&5&5&7 \end{array} \xrightarrow{\text{STD}} \begin{array}{cccccccc} 1&2&5&6&7&10&& \\ &&3&4&\boxed{x}&8&9&11 \end{array}$$

$\downarrow$ JDT move

$$\begin{array}{cccccccc} 1&2&5&6&\boxed{x}&10&& \\ &&3&4&7&8&9&11 \end{array}$$

Reading word: 1 2 5 6 7 10 3 4 8 9 11

$\sim$ 1 2 5 6 7 3 10 4 8 9 11

Move 3 past 7

$\sim$ 1 2 5 6 3 7 10 4 8 9 11

$\sim$ 1 2 5 6 3 7 4 10 8 9 11 Move 4 past 7

$\sim$ 1 2 5 3 6 7 4 10 8 9 11

$\sim$ 1 2 5 3 6 4 7 10 8 9 11 move 10 past 7

move the 1 out of the way → $\sim$ 1 2 5 3 6 4 10 7 8 9 11

$\sim$ 1 2 5 6 3 4 10 7 8 9 11 ← get 3, 4, 10 together

$\sim$ 1 2 5 6 3 10 4 7 8 9 11 move 10 past 3, 4

$\sim$ 1 2 5 6 10 3 4 7 8 9 11

QED

Def. The *rectification* of a skew SSYT is formed by performing inner slides until the tableau is a *straight* (non-skew) shape.

Note: Rectification is well-defined, i.e. order in which we perform the slides doesn't matter, because by above lemma it must be $= \mathrm{ins}(\mathrm{rw}(T))$.

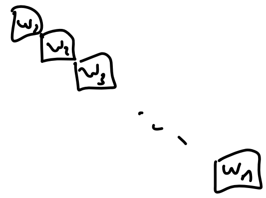
Ex:

$$\begin{array}{ccc} 2 & 4 & \\ \ast & 1 & 5 \\ & & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ 1 & 4 & 5 \\ & \ast & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ 1 & 4 & \\ \ast & 3 & 5 \end{array} \to \begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}$$

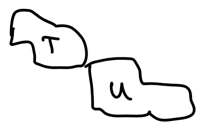
vs

$$\begin{array}{ccc} 2 & 4 & \\ & 1 & 5 \\ & \ast & 3 \end{array} \to \begin{array}{ccc} 2 & & \\ \ast & 4 & 5 \\ & 1 & 3 \end{array} \to \begin{array}{ccc} 2 & 4 & 5 \\ \ast & 1 & 3 \end{array} \to \begin{array}{ccc} 2 & 4 & \\ 1 & 3 & 5 \end{array}$$

Same!

Cor: Rectification of 

is $\mathrm{ins}(w)$.

Def: $T * U = \mathrm{rect}\Big($  $\Big)$

Cor: $(T * U) * S = T * (S * U)$

Define $*$ on words by considering their diagonal tableaux as above, and the reading word of the rectification.

So $w * v = \mathrm{rw}(\mathrm{ins}(w) \xleftarrow{\mathrm{ins}} v)$

The structure $(\text{Words}/_{\text{knuth equiv}}, *)$ is called the plactic monoid. The operation $*$ is actually just concatenation of representatives.

$\boxed{\text{Next time: Skew Schur functions}}$