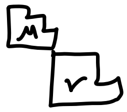
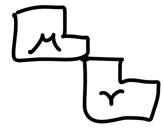
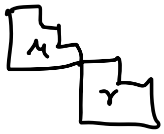
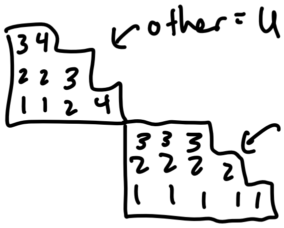
Products of Schur functions

Note: $s_\mu \cdot s_\nu = s$  ← skew shape formed by

$$= \sum c^{\rho}_{R\lambda}\, s_\lambda$$ where $\rho/R =$ .

Here, $c^{\rho}_{R\lambda} = \#$ LR tableaux of shape  and content λ.

Note that in any such tableau, the ν part is T_ν since ν is a straight shape:

$T =$ 

Want to show: $c^{\rho}_{R\lambda} = c^{\lambda}_{\mu\nu}$.

Method: Find a bijection btwn the T's above and skew tabs of shape λ/μ, content ν.

Bijection: First de-RSK (T_ν, T_ν):

$$\left(\begin{matrix}333\\2222\\11111\end{matrix},\ \begin{matrix}333\\2222\\11111\end{matrix}\right) \longleftrightarrow \left(\begin{matrix}1&2&3&3&3&1&2&2&2&1&1&1\\1&1&1&1&1&2&2&2&2&3&3&3\end{matrix}\right)\begin{matrix}\leftarrow r\\ \leftarrow b\end{matrix}$$

top row r is Knuth equiv to $rw(T_\nu)$, so inserting it into U also gives highest weight of content/shape λ.

Insert r into u and label new squares w/ b; this forms a skew SSYT of content r on top of and shape λ/μ:

34
223
1124 ← 123331222111

Recorded skew:

1

4
33
222
1114 ← 2333122211

Recorded skew SSYT.

4
33
2224
1112 ← 333122211

13
233
1222
111

Littlewood-Richardson!
(HWK)

4
33
2224
1112333 ← 122211

4
334
2222333
1111222 ← 111

44
33333
2222222
1111111

□

Cor: $c^{\lambda}_{\mu\nu} = c^{\lambda}_{\nu\mu}$ (direct combinatorial pf to be assigned on HW 6)

Pf: $s_\mu s_\nu = s_\nu s_\mu$. □

Crash course in representation theory

Def: A group $(G,*)$ is a set G w/ a binary operation $*: G\times G \to G$ with:

- Identity: $\exists e\in G$, $e*g = g*e = g$ for all g
- Associativity: $g*(h*j) = (g*h)*j$
- Inverses: For all $g\in G$, $\exists h\in G$, $gh = hg = e$

Def: A representation of a finite group G

Ex: $(S_n, \circ)$ $\quad (GL_n(\mathbb{C}), \cdot)$

$(\mathbb{Z}, +)$

Def: A n-dim'l (matrix) representation of G is an assignment

$$g \mapsto M_g$$

of an $n\times n$ matrix $M_g \in GL_n(\mathbb{C})$ to every $g\in G$, s.t.

$$M_g \cdot M_h = M_{g*h}.$$

Faithful if every group elt maps to a unique matrix.

Ex: In S_3, map each elt to 1×1 matrix of its sign

$\mathrm{id} \mapsto (1)$

$(12) \mapsto (-1)$

$(13) \mapsto (-1)$

$(23) \mapsto (-1)$

$(123) \mapsto (1)$

$(132) \mapsto (1)$

<u>Ex:</u> Permutation representation:

$$\mathrm{id} \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad (12)\cdot(23) = (123)$$

$$(12) \mapsto \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} \qquad \begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

✓

$$(13) \mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$(23) \mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix}$$

$$(123) \mapsto \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}$$

$$(321) \mapsto \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}$$

<u>Def:</u> Two representations $g \mapsto M_g$ and $g \mapsto N_g$ are <u>isomorphic</u> if $\exists\, A \in GL_n(\mathbb{C})$,

$$A\, M_g A^{-1} = N_g \text{ for all } g.$$

Can also think of a rep as a vector space $V \cong \mathbb{C}^n$ w/ an action $G \times V \to V$ by linear maps. (Basis-free)

Operations on representations

① $\oplus$ (Direct sum)

$V \oplus W$, inherit group action componentwise

In terms of matrices: $g \mapsto \left(\begin{array}{c|c} M_g & \\ \hline & N_g \end{array}\right)$

Ex: Permutation rep of S_3 is isomorphic to a direct sum:

- Each matrix fixes $\begin{pmatrix}1\\1\\1\end{pmatrix}$, so make that a basis vector along w/ basis $\begin{pmatrix}1\\-1\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\-1\end{pmatrix}$ for orthogonal space:

$A = \begin{pmatrix}1 & 1 & 0\\ 1 & -1 & 1\\ 1 & 0 & -1\end{pmatrix}$ sends $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}$ to $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix}$.

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ 1 & 1 & -2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 2 & -1 & -1\\ 1 & 1 & -2\end{pmatrix}$$

$$A^{-1} I A = \left(\begin{array}{c|cc} 1 & & \\ \hline & 1 & \\ & & 1 \end{array}\right)$$

$$A^{-1}\begin{pmatrix} & 1 & \\ 1 & & \\ & & 1 \end{pmatrix} A = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 2 & -1 & -1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & -1 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

etc (finish for hwk)

<u>Def</u>: A representation is <u>irreducible</u> if it is not the direct sum of two subrepresentations.

<u>Thm</u> (Maschke/Schur): Every rep is uniquely (up to isom) the direct sum of irreducibles.

<u>Thm (next time)</u> The irreducible reps of S_n

$\longleftrightarrow$ partitions of n.

② <u>$\otimes$ (tensor product)</u> V dim n, W dim m

$V \otimes W$ = vector space of dim $n \cdot m$
generated by symbols $v \otimes w$ $(v \in V, w \in W)$,
mod relations:

$$c(v \otimes w) = (cv) \otimes w = v \otimes (cw)$$
$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$
$$v \otimes (w_1 + w_2) = (v \otimes w_1) + (v \otimes w_2)$$

If g acts on V and on W,
it acts on $V \otimes W$ by (inner tensor product)

$$g(v \otimes w) = (M_g \cdot v) \otimes (N_g \cdot w)$$

Matrix operation: $A \otimes B = \left(\begin{array}{c|c|c} a_{11}B & a_{12}B & \cdots \\ \hline a_{21}B & \ddots & \\ \hline & & a_{nn}B \end{array}\right)$

Ex: $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 0 & 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix}$

(3) Induced representations: Suppose $H \subseteq G$ subgroup
(ex: $S_3 \subseteq S_4$, $S_3 \times S_2 \subseteq S_5$).

Let V be a rep of H. Then

$$\mathrm{Ind}\uparrow_H^G V = \mathbb{C}G \otimes_{\mathbb{C}H} V$$ ← "H-module tensor product"

$$= \{ (\alpha_1 g_1 + \alpha_2 g_2 + \dots + \alpha_r g_r) \otimes v \} \Big/ \left(h(g \otimes v) = (hg) \otimes v = g \otimes hv \right)$$

$\underline{\text{Ex}}$: Take triv. rep of S_2: $\left.\begin{array}{l}(id) \mapsto (1), \\ (12) \mapsto (1)\end{array}\right\} \mathbb{1}_{S_2}$

v_0 basis

induce to S_3:

$\mathrm{Ind}_{S_2}^{S_3} \mathbb{1}_{S_2}$ spanned by coset reps

$$\left.\begin{array}{l} v_1 = 123 \otimes v_0 \\ v_2 = 132 \otimes v_0 \\ v_3 = 312 \otimes v_0 \end{array}\right\} \begin{array}{l}\text{list notation:} \\ \text{others obtained} \\ \text{by applying } (12)\end{array}$$

1 to the left of 2 in each.

S_3 action: in v_1, v_2, v_3 basis:

$$id \mapsto \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \qquad (13) \mapsto \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix}$$

$$(12) \mapsto \begin{pmatrix} 1 & & \\ & & 1 \\ & 1 & \end{pmatrix} \qquad (23) \mapsto$$

$\therefore$ permutation rep!

$\underline{\text{Def}}$: The <u>outer</u> <u>tensor</u> <u>product</u> of a rep V of S_n w/ a rep W of S_m is

$$\mathrm{Ind}_{S_n \times S_m}^{S_{n+m}} V \otimes W$$

where $V \otimes W$ is a rep of $S_n \times S_m$ by $(g,h)\cdot(v \otimes w) = gv \otimes hw$.

Construction of irreducible representations of S_n

Fact from rep thy: # irred reps of G
= # conjugacy classes of G

- # Conj classes of S_n = # cycle types $= p(n)$

Construction of irred. reps of S_n: via Garnir polynomials (in other books: "Young tabloids", "Specht modules". This will be equivalent).

V_λ = irred rep of S_n indexed by $\lambda \vdash n$.

Notice: S_n acts on $\mathbb{C}[x_1, \ldots, x_n]$ (polynomial ring) by permuting the variables.

V_λ is a sub-rep of this polynomial vector space.

Def: Let T be an SYT of shape λ.

The Garnir polynomial F_T is

$$F_T = \prod_{\substack{i<j \\ \text{in same} \\ \text{column}}} (x_i - x_j)$$

Def: $V_\lambda = \mathrm{Sp}\{F_T : T \in SYT(\lambda)\} \subseteq \mathbb{C}[x_1, \ldots, x_n]$

Ex: $V_{\ydiagram{3}} = sp\underset{\boxed{1\,2\,3}}{\{1\}}$ = trivial rep

$V_{\ydiagram{2,1}} = sp\{\underset{\begin{smallmatrix}2\\1&3\end{smallmatrix}}{x_2-x_1},\ \underset{\boxed{\begin{smallmatrix}3\\1&2\end{smallmatrix}}}{x_3-x_1}\}$

$V_{\ydiagram{1,1,1}} = sp\{\underset{\boxed{\begin{smallmatrix}3\\2\\1\end{smallmatrix}}}{(x_3-x_2)(x_3-x_1)(x_2-x_1)}\}$ = sign rep.

Ex: $V_{\ydiagram{2,2,1}} = sp\{(x_3-x_2)(x_3-x_1)(x_2-x_1)(x_5-x_4),$ $\quad\boxed{\begin{smallmatrix}3\\2&5\\1&4\end{smallmatrix}}$

$(x_4-x_2)(x_4-x_1)(x_2-x_1)(x_5-x_3),$ $\quad\begin{smallmatrix}4\\2&5\\1&3\end{smallmatrix}$

$\vdots$ $\quad\begin{smallmatrix}5\\2&4\\1&3\end{smallmatrix}$

$\quad\begin{smallmatrix}4\\3&5\\1&2\end{smallmatrix}$

$\}$ $\quad\begin{smallmatrix}5\\3&4\\1&2\end{smallmatrix}$

Ex: Let's analyze S_3 action on $V_{\ydiagram{2,1}}$:

$$x_2-x_1 = \begin{pmatrix}-1\\1\\0\end{pmatrix} \qquad x_3-x_1 = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

S_3 acts by permutation matrices
$\Rightarrow$ it's the 2D irred. component of the

permutation rep.

<u>Cor</u>: Permutation rep of $S_3 = V_{\ydiagram{3}} \oplus V_{\ydiagram{2,1}}$

<u>Lemma</u>: Permutation rep of $S_n = V_{\underbrace{\ydiagram{4}}_{n}} \oplus V_{\ydiagram{3,1}}$

<u>Pf</u>: Hwk

<u>Project suggestion</u>: Action of S_n on <u>coinvariant</u> <u>ring</u>

$$\mathbb{C}[x_1, \dots, x_n] / (e_1, \dots, e_n),$$

decomposition into irreducibles, maj formula, higher Specht (garnir) polynomials

---

Project suggestion:

<u>Kronecker</u> <u>coefficients</u> are the coefficients in the expansion

$$V_\nu \otimes V_\mu = \sum_\lambda g^\lambda_{\mu\nu} V_\lambda .$$

both S_n reps, inner tensor product

Find a combinatorial formula for $g^\lambda_{\mu\nu}$ (OPEN)

Connecting S_n reps to Schur functions

Main Fact (will not prove in this class)

$$\mathrm{Ind}\uparrow_{S_k \times S_{n-k}}^{S_n} V_\mu \otimes V_\nu = \bigoplus c_{\mu\nu}^\lambda V_\lambda$$

$c_{\mu\nu}^\lambda$ copies of V_λ.

This allows us to define:

Def: The Frobenius map Frob sends a representation $V = \oplus c_\lambda V_\lambda$ to the symmetric function $f = \sum c_\lambda s_\lambda$.

It sends the outer tensor product to multiplication:

$$V_\mu \otimes_{out} V_\nu \longrightarrow s_\mu s_\nu$$

$$\mathrm{Frob}(V \otimes_{out} W) = \mathrm{Frob}(V) \cdot \mathrm{Frob}(W).$$

Def: Can extend to virtual representations: linear combinations $\sum c_\lambda V_\lambda$ where c_λ's are not nec. positive integers, in $\mathbb{Q}$.
Then Frob is a ring hom. from the ring of

virtual reps of all S_n's under $\oplus$ and $\otimes_{out}$,
to Λ.

Cor: A symm function is Schur positive iff it is the Frob image of a representation.

Ex: Decompose $V_{\square\square\square} \otimes V_{\square\square\square}$ into irreducible representations. What is its Frobenius character? In terms of h basis?

[Monday: go over ballot problem first]

Garnir relations: To show V_λ is an irreducible S_n-module:

1. Actually define it as $sp(F_T : T \in Tab(\lambda))$
   general filling w/ $1, \ldots, n$

2. This space is S_n-invariant in $\mathbb{C}[x_1, \ldots, x_n]$ because $\pi F_T = F_{\pi T}$

3. Garnir Relations give a "straightening" algorithm for expressing a general F_T in terms of SYT F_T's:

Lemma 1 (column straightening). Given a filling T of λ with the letters $1, 2, \ldots, n$, let T' be the tableau formed by reordering the letters within each column of T from least to greatest. Then

$$F_{T'} = \pm F_T .$$

Pf: Transposing two letters in the same column negates the Vandermonde determinant in that column and therefore negates F_T. Therefore, if $T' = \pi T$,

$$F_{T'} = \operatorname{sgn}(\pi) F_T .$$

Ex: $T = \begin{array}{llll} 8 & 7 & & \\ 1 & 2 & 5 & \\ 4 & 9 & 3 & 6 \end{array} \rightsquigarrow T' = \begin{array}{llll} 8 & 9 & & \\ 4 & 7 & 5 & \\ 1 & 2 & 3 & 6 \end{array}$

$\pi = (14)(2\,7\,9)$

$$F_T = (x_8 - x_1)(x_8 - x_4)(x_1 - x_4) \cdot (x_7 - x_2)(x_7 - x_9)(x_2 - x_9) \cdot (x_5 - x_3)$$

$$F_{T'} = (x_8 - x_1)(x_8 - x_4)(x_4 - x_1)(x_7 - x_2)(x_9 - x_7)(x_9 - x_2)(x_5 - x_3)$$

$$= (-1)^3 F_T = -F_T$$

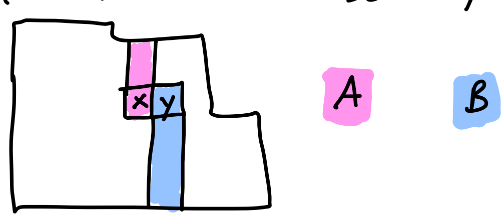
Lemma 2: (Column exchanges for row straightening):

Suppose T is column-increasing but not row-increasing.

Choose the topmost row having a decrease, and the rightmost decrease in that row, $\boxed{x|y}$, $x>y$.

Let A = set of squares weakly above x in its col

B = set " " " below y in its col.



Define the <u>Garnir operator</u> $$g_{A,B} = \sum_{\substack{\sigma \in S_{A \cup B} \\ \text{that preserves} \\ \text{col increasing} \\ \text{within } A, B}} \mathrm{sgn}(\sigma)\cdot\sigma .$$

Then $g_{A,B} F_T = \sum \mathrm{sgn}(\sigma) F_{\sigma T} = 0.$

Ex: $T = \begin{array}{|c|c|c|c|} \hline 8 & 9 \\ \hline 4 & 7 & 5 \\ \hline 1 & 2 & 3 & 6 \\ \hline \end{array}$ $\quad g_{A,B} = id - (5\,7) + (3\,7)(5\,9) + (5\,9\,7) + (3\,5\,7) - (3\,5\,9\,7)$

$$\Rightarrow F_T = F_{\begin{array}{cccc} 8 & 9 \\ 4 & 5 & 7 \\ 1 & 2 & 3 & 6 \end{array}} - F_{\begin{array}{cccc} 8 & 5 \\ 4 & 3 & 9 \\ 1 & 2 & 7 & 6 \end{array}} - F_{\begin{array}{cccc} 8 & 7 \\ 4 & 5 & 9 \\ 1 & 2 & 3 & 6 \end{array}} - F_{\begin{array}{cccc} 8 & 9 \\ 4 & 3 & 7 \\ 1 & 2 & 5 & 6 \end{array}} + F_{\begin{array}{cccc} 8 & 7 \\ 4 & 3 & 9 \\ 1 & 2 & 5 & 6 \end{array}}$$

all row-increasing at $\boxed{x|y}$ ✓

Can focus on those two cols:

$(x_9 - x_7)(x_9 - x_2)(x_7 - x_2)(x_5 - x_3) = \ldots$

Pf sketch (by exs)

- Focus on a single monomial btwn the two columns, show it occurs w/ multiplicity 0:

e.g. $x_7^2 x_5 x_2$ $\rightsquigarrow$ swap the 9 and 3, change sign

$$\begin{array}{|c|c|} \hline 7 & \\ \hline 3 & 9 \\ \hline 2 & 5 \\ \hline \end{array} \quad \text{vs} \quad \begin{array}{cc} 9 & \\ 7 & 5 \\ 2 & 3 \end{array}$$

(swapped and then straightened both cols $\Rightarrow$ neg sign)

In general each valid monomial will appear twice, w/ opposite signs:

$x_9^2 x_7 x_3$ (no x_2):

$$-- \begin{array}{|c|c|} \hline 9 & \\ \hline 7 & 5 \\ \hline 2 & 3 \\ \hline \end{array} \qquad - \begin{array}{|c|c|} \hline 9 & \\ \hline 3 & 7 \\ \hline 2 & 5 \\ \hline \end{array} \qquad \checkmark$$

+ sign change

$\square$

Why irreducible? Any given F_T generates all other $F_{\pi T}$'s by S_n action, so suffices to show that any S_n-invariant subspace of V_λ contains some F_T.

<u>Claim:</u> $\displaystyle\sum_{\substack{\pi \in Col(T') \\ \uparrow \\ \text{column permutations}}} \mathrm{sgn}(\pi) \cdot \pi F_T = \pm F_{T'}$.

<u>Pf:</u> (in class)

Cor: Using $\sum_{\pi \in Col(T)} sgn(\pi) \cdot \pi$ applied to any elt of a subspace W, get a constant times F_T, and hence the whole module.

Cor: V_λ is irreducible.