Grassmannians and flag varieties over $\mathbb{R}$ or $\mathbb{C}$

Topology of projective space (over $\mathbb{R}$ or $\mathbb{C}$)

Ex: $\mathbb{P}^2 = \{(1:x:y)\} \cup \{(x:1:y)\} \cup \{(x:y:1)\}$

(not disjoint union)

$= \mathbb{R}^2 \cup \mathbb{R}^2 \cup \mathbb{R}^2$

Each is called an affine patch. Each $\mathbb{R}^2$ has its usual topology, glue together w/: a set is open in $\mathbb{P}^2$ if its intersection w/ every affine patch is open.

Limits:

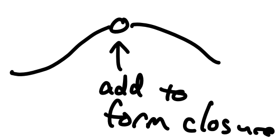
$$\lim_{t\to\infty} (1:t) = \lim_{t\to\infty} \left(\tfrac{1}{t}:1\right) = \lim_{x\to 0} (x:1) = (0:1)$$

change patch

$$\lim_{t\to\infty} (1:t^2:t) = (0:1:0)$$

$$\lim_{t\to\infty} (1:2t:3t) = (0:2:3)$$

Closures: add all limit points to a set



Ex: $\overline{\{(0:0:1:x:y:z)\}}$ includes

$\{(0:0:0:1:y:z)\}$, $\{(0:0:0:0:1:z)\}$, and $\{(0:0:0:0:0:1)\}$

Nothing else: Complement is all pts having one of first two coords nonzero

= union of two (open) affine patches

$(1 : x : \cdots)$ and $(x : 1 : \cdots)$

= open. (Same over $\mathbb{C}$).

<u>Topology of $Gr_{\mathbb{R}}(k,n)$:</u>

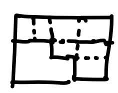
Patches are similar, when some $k \times k$ minor is fixed to be the identity $\rightsquigarrow$ copy of $\mathbb{R}^{k(n-k)}$, glue together topologies.

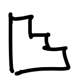
<u>Limits:</u> What is $\overline{\Omega^\circ_\lambda}$?

<u>Thm:</u> $\overline{\Omega^\circ_\lambda} = \bigcup_{\mu \supseteq \lambda} \Omega^\circ_\mu$ $\leftarrow$ write $X_\lambda = \overline{\Omega^\circ_\lambda}$, called <u>Schubert variety</u>

<u>Pf:</u> First we'll show any such Ω°_μ is in the closure, by example:

$$\begin{pmatrix} 1 & 2 & 3 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 \end{pmatrix} \in \Omega^\circ$$



Show we can limit to a point in Ω°  or Ω°  or Ω  :

each of these coverings $\lambda \to \mu$ is pushing a pivot 1 step to the right.

- $$\begin{pmatrix} 1 & t & xt & 0 & yt & 0 & zt \\ & & & 1 & w & 0 & r \\ & & & & & 1 & s \end{pmatrix} \xrightarrow[t\to\infty]{\lim} \begin{pmatrix} 0 & 1 & x & 0 & y & 0 & z \\ & & & 1 & w & 0 & r \\ & & & & & 1 & s \end{pmatrix}$$

- $$\begin{pmatrix} 1 & x & y & & -zt & 0 & w+rz \\ & & & 1 & t & 0 & tr \\ & & & & & 1 & s \end{pmatrix} \sim \begin{pmatrix} 1 & x & y & z & 0 & 0 & w \\ & & & 1 & t & 0 & tr \\ & & & & & 1 & s \end{pmatrix}$$

$$\xrightarrow[t\to\infty]{} \begin{pmatrix} 1 & x & y & 0 & 0 & 0 & w \\ & & & 0 & 1 & 0 & r \\ & & & & & 1 & s \end{pmatrix}$$

- $$\begin{pmatrix} 1 & x & y & & z & & -tr \\ & & & 1 & w & & -ts \\ & & & & & 1 & t \end{pmatrix} \sim \begin{pmatrix} 1 & x & y & & z & r & \\ & & & 1 & w & s & \\ & & & & & 1 & t \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & x & y & & z & r & \\ & & & 1 & w & s & \\ & & & & & & 1 \end{pmatrix}$$

So closure includes all larger partitions. ✓

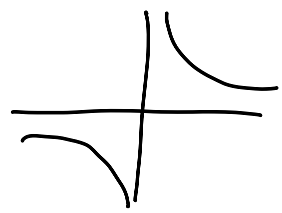
Why nothing else? Complement is open: union of "affine patches" in which some det to the left of the λ pivots is nonzero. □

(Hwk: Plücker coordinates).

Rational equivalence and cohomology/chow ring

Variety: a set of points defined by polynomial equations (projective variety: in $\mathbb{P}^n$).

Ex: $xy = z^2$



Ex: $(x:y:z:w)$ $\mathbb{P}^3$ coordinates

$z=0,\ w=0$ define a line in $\mathbb{P}^3$

Rational equivalence: Two varieties V, W are rationally equivalent if there are parameterized families of polynomials

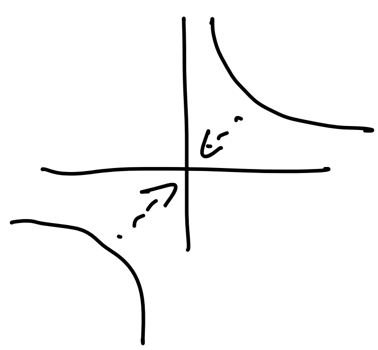
$$f_i(x_1, \dots, x_n; t)$$

s.t. $$\bigcap \big(f_i(x_1, \dots, x_n; 0)\big) = V,$$

$$\bigcap \big(f_i(x_1, \dots, x_n; 1)\big) = W.$$

Ex: $xy = tz^2$

So a hyperbola is rationally equivalent to two lines.



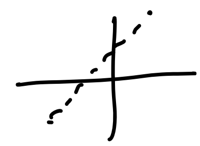
Def: Chow ring: of a variety is ring of subvarieties (defined by further equations) up to rational equivalence,

Written $A^*(V) = \bigoplus_d \underbrace{A^d(V)}_{\text{classes of codimension } d}$

where the $+$ in the ring is union
the $\cdot$ is intersection of two "generic" elements.

Ex: Let σ = class of a line in $\mathbb{P}^2$
μ = class of $(xy = z^2)$ in $\mathbb{P}^2$

Then $\mu = 2\sigma$



and $\mu \cdot \sigma$ = class of 2 points
$\sigma^2 = [pt]$

→ σ generates $A^1(\mathbb{P}^2)$ (why? Hwk)

→ σ^2 generates $A^2(\mathbb{P}^2)$

→ Class of $\mathbb{P}^2$ (written 1) generates $A^0(\mathbb{P}^2)$

$$A^*(\mathbb{P}^2) = A^0 \oplus A^1 \oplus A^2 \cong \underbrace{\mathbb{C}[\sigma]}_{\text{gen by } 1, \sigma, \sigma^2.} / (\sigma^3)$$

Also called cohomology ring $H^*(\mathbb{P}^2)$
(agrees w/ A^* in all cases considered here)

Grassmannian:

Thm: $A^*(Gr(k,n)) \cong \Lambda(x_1, x_2, \dots) \big/ \left(s_\lambda : \lambda \not\subseteq k\,\underbrace{\boxed{}}_{n-k}\right)$.

Basis: $[X_\lambda] =: \sigma_\lambda \mapsto s_\lambda$

Other elts of $[X_\lambda]$ include Schubert varieties wrt different "flags" (basis change)

Basis-free def: Consider standard flag given by $\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{pmatrix} = F_\bullet$.

Then $X_\lambda = X_\lambda(F_\bullet) = \{ V \in Gr(k,n) \mid \dim(V \cap F_{n-k+i-\lambda_i}) \geq i \}$

$$\begin{pmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{pmatrix}$$

$X_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}$: $\dim V \cap F_2 \geq 1$

$\dim V \cap F_3 \geq 2$

Can change flag to whatever we want.

Ring structure: $[X_\lambda] \cdot [X_\mu] = [X_\lambda(F_\bullet) \cap X_\mu(G_\bullet)]$

for transverse flags F, G

$= \sum c_{\lambda\mu}^{\nu} [X_\nu]$!!

Ex: 2 lines through 4 lines in 3D space b/c $\sigma_{\square}^4 = 2\sigma_{\boxplus}$.